Math 518: HW 13 due Wednesday, December 10, 2014.

Note: My office hour on Friday, December 5 is cancelled.

Final Exam: Monday, December 15 from 8–11am in our usual classroom. You may bring two sheets of paper (writing/printing on all four sides is permitted) to the exam.

1. Use the Mayer-Vietoris sequence to compute H^* of the following spaces. Here, you may take as given only the computations of H^* done in class; specifically, you should not use Theorems 17.31 or 17.32 of Lee. Hint: You may need to apply the Mayer-Vietoris sequence multiple times. E.g. for (a) first do the case of $\mathbb{R}^2 \setminus \{\text{two points}\}$.

   (a) $\mathbb{R}^2 \setminus \{\text{three points}\}$.

   (b) $T = S^1 \times S^1$.

   (c) $S^1 \times S^2$.

2. The Classification of Surfaces says (in part) that any smooth orientable 2-manifold without boundary is diffeomorphic to exactly one of $S^2, T = S^1 \times S^1, T\#T, T\#T\#T, \ldots$ where $\#$ denotes the connected sum operation defined on page 225 of Lee.

   Prove the "exactly one" part of this: Specifically, show that no two surfaces on this list are diffeomorphic by calculating that $S_g = \#_{i=1}^{g} T$ has $H^1 = \mathbb{R}^{2g}$. (By convention, $S_0 = S^2$.)

3. (a) Suppose M^n is the union of open sets U and V where $H^*(U), H^*(V)$, and $H^*(U \cap V)$ are all finite-dimensional. Prove that $H^*(M)$ is also finite-dimensional.

   (b) Fun Fact: Suppose M is a smooth n-manifold without boundary. Then one can cover M by open sets $\{U_i\}$ so that each U_i is diffeomorphic to $\mathbb{R}^n$, each U_i has compact closure, any finite intersection of the U_i is either empty or diffeomorphic to $\mathbb{R}^n$, and finally any point of M is in only finitely many U_i.

   Your task is to show directly that this can be done for $\mathbb{R}^2$.

   Note: The proof in general uses a Riemannian metric so one can talk about convex sets...

   (c) Use (a) and the Fun Fact to prove that any compact smooth M without boundary has finite-dimensional $H^*(M)$.

4. Suppose a smooth map $F \colon S^n \to S^n$ is not onto. Prove that $\deg(F) = 0$. (This is really just a step of Problem 5.)

5. Let p be a nonconstant polynomial with complex coefficients $p \in \mathbb{C}[z]$. View $\mathbb{CP}^1$ as $\mathbb{C} \cup \{\infty\}$ where $\mathbb{C}$ is the points with projective coordinates $[z : 1]$ and ∞ is the point $[1 : 0]$. Define $\widetilde{p} \colon \mathbb{CP}^1 \to \mathbb{CP}^1$ by $\widetilde{p}(z) = p(z)$ for $z \in \mathbb{C}$ and $\widetilde{p}(\infty) = \infty$.

   (a) Prove that $\widetilde{p}$ is smooth.

   (b) Use the behavior of $\widetilde{p}$ near ∞ to prove that the topological degree $\deg(\widetilde{p})$ is the algebraic degree of the polynomial p.

   (c) Combine (b) with Problem 4 to prove the Fundamental Theorem of Algebra: any nonconstant polynomial in $\mathbb{C}[z]$ has a root in $\mathbb{C}$.