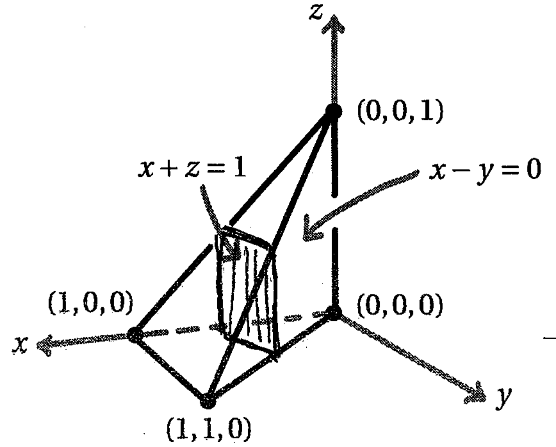
1. Fill in the limits and integrand of the triple integral below so that it computes the volume of the tetrahedron shown at right. **(4 points)**



Other correct answers

$$\int_0^1 \int_0^{1-z} \int_y^{1-z} 1 \, dx\,dy\,dz$$

$$\int_0^1 \int_0^{1-z} \int_0^{x} 1 \, dy\,dx\,dz$$

$$\int_0^1 \int_y^1 \int_0^{1-x} 1 \, dz\,dx\,dy$$

$$\int_0^1 \int_0^{1-y} \int_y^{1-z} 1 \, dx\,dz\,dy \qquad \int_0^1 \int_0^{1-x} \int_0^{x} 1 \, dy\,dz\,dx$$

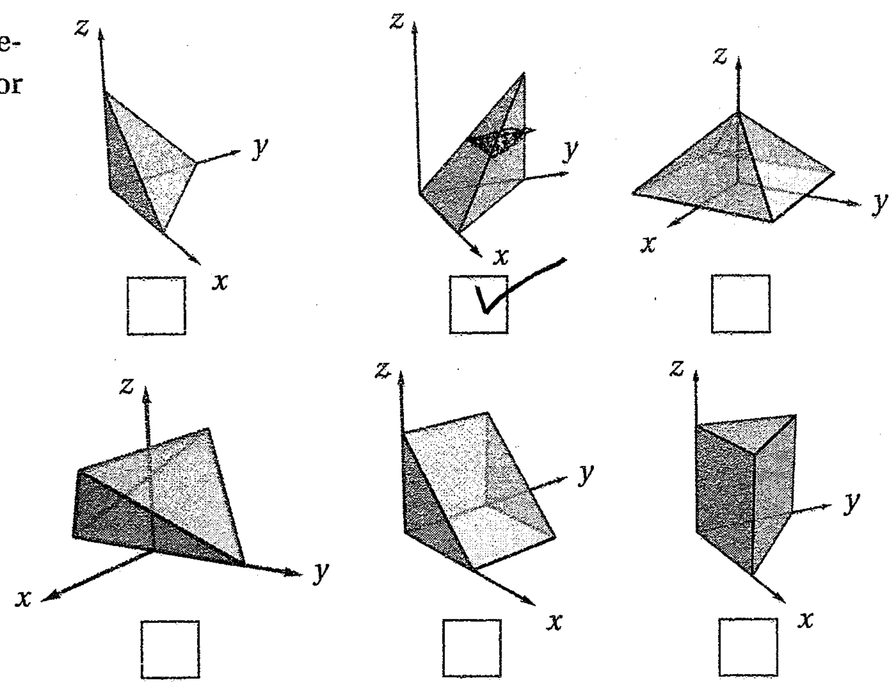
slice with x fixed is a rectangle.

$$\text{Volume} = \int_0^1 \int_0^{x} \int_0^{1-x} 1 \; dz\,dy\,dx$$

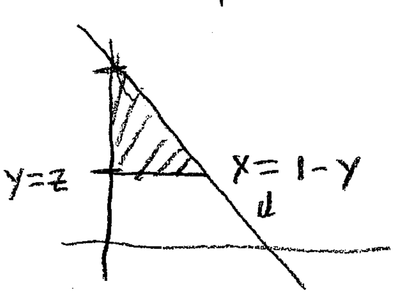
2. Mark the box below the picture corresponding to the region of integration for the triple integral:

$$\int_0^1 \int_z^1 \int_0^{1-y} f(x,y,z)\, dx\,dy\,dz$$

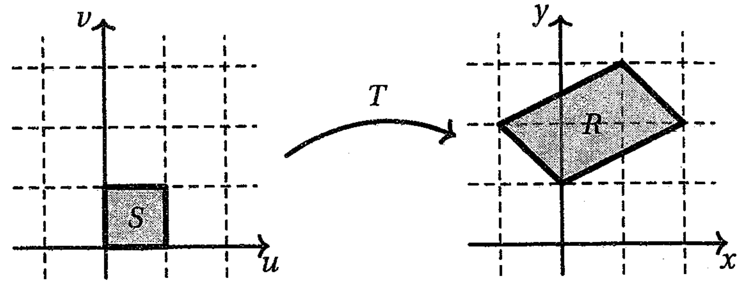
(2 points)



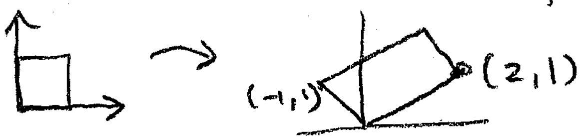
Slice by z



3. Find a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ taking the unit square S to the parallelogram R shown at right, where both are shown against a grid of unit squares. **(4 points)**



First do a linear transf



corresp to $\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$, which is $L(u,v) = (-u+2v,\ u+v)$. Follow this by shifting up by 1

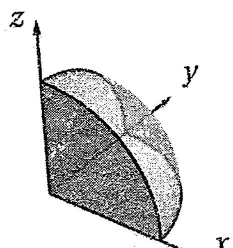
Also correct $(-v+2u,\ u+v+1)$

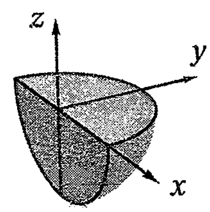
$T(u,v) = (\ -u+2v\ ,\ u+v+1\ )$

4. For each of the given integrals, label the box below the picture of the corresponding region of integration in spherical coordinates. **(2 points each)**

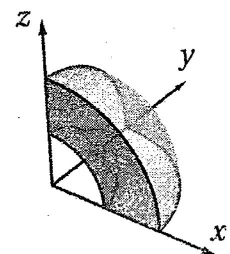
(A) $\displaystyle\int_{\pi/2}^{\pi}\int_{0}^{\pi/2}\int_{1}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

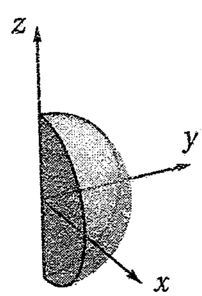
(B) $\displaystyle\int_{0}^{\pi}\int_{\pi/2}^{\pi}\int_{0}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

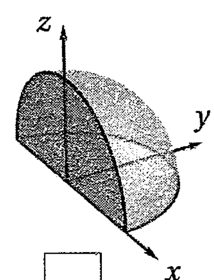


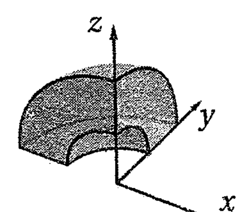


B





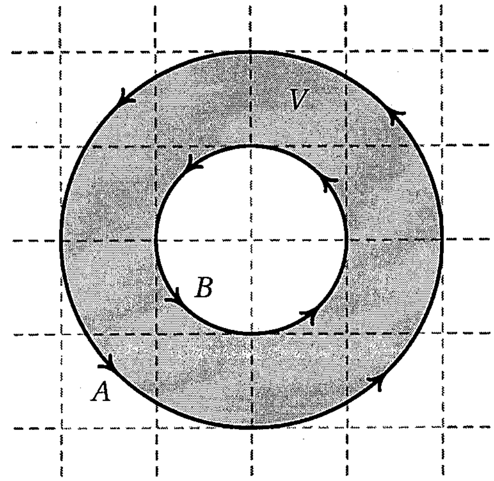




A

Scratch Space

5. Let A and B be the oriented circles shown at right against a grid of unit squares, and let V be the region between them. Suppose $F = \langle P, Q\rangle$ is a vector field on V where



$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y} - 3 \quad \text{and} \quad \int_A \mathbf{F}\cdot d\mathbf{r} = 10.$$

Compute $\int_B \mathbf{F}\cdot d\mathbf{r}$.

Note $\partial V = A - B$ when oriented per Green. So as

$$\iint_V \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint_V -3\, dA = 3 \cdot \text{Area}$$

$$= -3(\pi 2^2 - \pi 1^2) = -9\pi$$

We know

$$9\pi = \int_{\partial V} \vec{F}\cdot d\vec{r} = \int_A \vec{F}\cdot d\vec{r} - \int_B \vec{F}\cdot dr$$

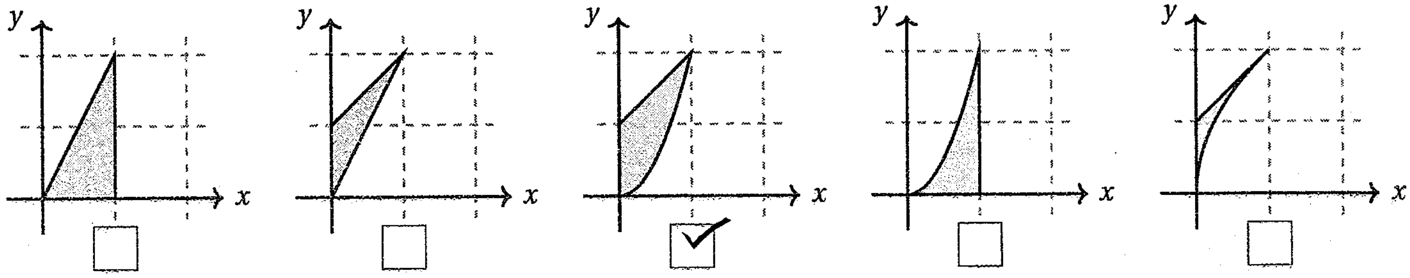
$$= 10 - \int_B \vec{F}\cdot d\vec{r} \implies \int_B \vec{F}\cdot d\vec{r} = 10 + 9\pi$$

$\int_B \mathbf{F}\cdot d\mathbf{r} =$ $10 + 9\pi$

Scratch Space

6. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation $T(u,v) = (v,\ u - v + 2uv)$. Let S be the triangle in the (u,v)–plane whose vertices are $(0,0), (1,0), (1,1)$ and let $R = T(S)$ be the region that is the image of S under T.

(a) Check the box below the picture of R drawn against a dashed grid consisting of **unit squares.** **(2 points)**

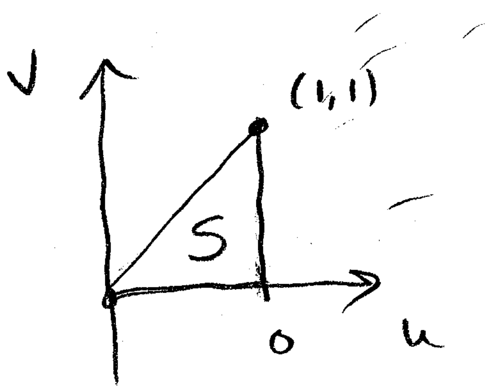


(b) Fill in the limits and integrand of the integral below so that it computes $\iint_R \cos(x+y)\,dA$ as an integral over the square S. **(4 points)**

$$J = \begin{pmatrix} 0 & 1 \\ 1+2v & -1+2u \end{pmatrix} \quad \text{has det} = -(1+2v)$$

So $|\det J| = 1 + 2v$

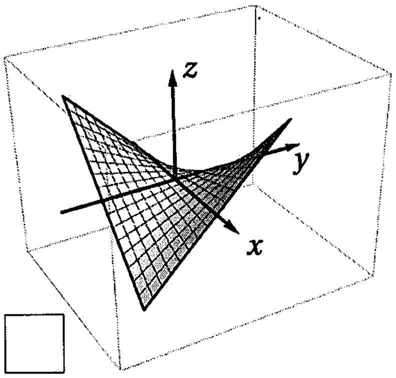
$$\cos(x+y) = \cos(u + 2uv)$$

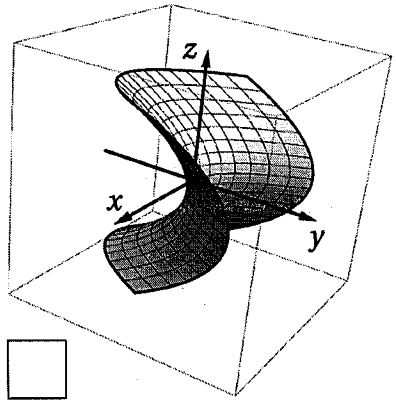


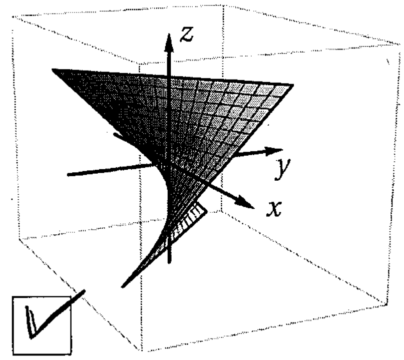
$$\iint_R \cos(x+y)\,dA = \int_0^1 \int_0^u \cos(u+2uv)\,(1+2v)\; dv\,du$$

7. Let S be the surface parameterized by $\mathbf{r}(u,v) = \langle u,\ uv,\ v\rangle$ for $-1 \le u \le 1$ and $-1 \le v \le 1$.

(a) Mark the picture of S below.







(b) Completely setup, but do not evaluate, the surface integral $\iint_S x^2\, dS$.

$$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & v & 0 \\ 0 & u & 1 \end{vmatrix} = (v, -1, u) \quad \text{so } dS = \sqrt{1+u^2+v^2}\, du\, dv$$

For (c) $\sqrt{1+u^2+v^2}$ has a min of 1 and a max of $\sqrt{3} \approx 1.7$ on $D = \left\{ \begin{matrix} -1 \le u \le 1 \\ -1 \le v \le 1 \end{matrix} \right\}$, so area is between 4 and $\sqrt{3}\cdot 4 \approx 6.8$. So must be 5.

$$\iint_S x^2\, dS = \int_{-1}^{1} \int_{-1}^{1} u^2 \sqrt{1+u^2+v^2}\; du\, dv$$

(c) Circle the number closest to the area of S: 1 3 5 7 9

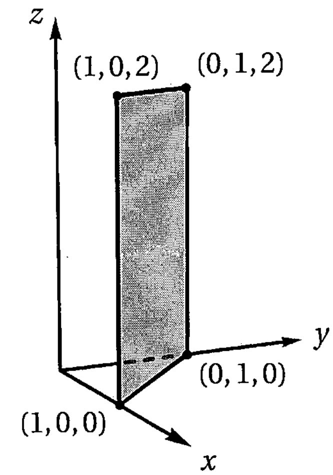
(d) Find the tangent plane to S at $(0,0,0)$. [You *must* show work that justifies your answer.]

normal at $(0,0,0)$ is $\vec{r}_u \times \vec{r}_v$ @ $u=0, v=0$ which is $(0,-1,0)$. So plane is

$$0\cdot(x-0) + (-1)(y-0) + 0\cdot(z-0) = 0.$$

Equation: $\boxed{0}\, x + \boxed{1}\, y + \boxed{0}\, z = \boxed{0}$

8. Let R be the rectangle in the plane $x + y = 1$ with vertices shown at right.



(a) Parameterize R by $\mathbf{r}\colon D \to \mathbb{R}^3$, being sure to specify the domain D of the parameterization in the (u, v)-plane.

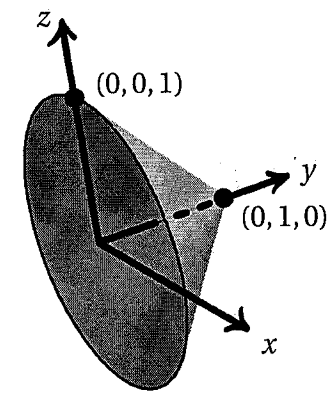
Use $u = x$ and $v = z$ as param,

and so $y = 1 - x$

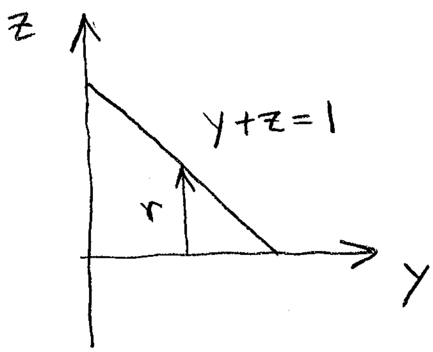
$D = \{0 \leq u \leq 1,\ 0 \leq v \leq 2\}$ $\quad \mathbf{r}(u, v) = \langle u, 1 - u, v \rangle$

(b) The integral $\iint_R z - 1\, dS$ is: negative **zero** positive

9. Consider the cone C shown at right.



(a) Parameterize C by $\mathbf{r}\colon D \to \mathbb{R}^3$, being sure to specify the domain D of the parameterization in the (u, v)-plane.



Params: $u = y$

and v = angle around z-axis

the "polar r" is a function of $u = y$, namely $r = 1 - u$

Also correct: $\langle u \cos v,\ 1 - u,\ u \sin v \rangle$

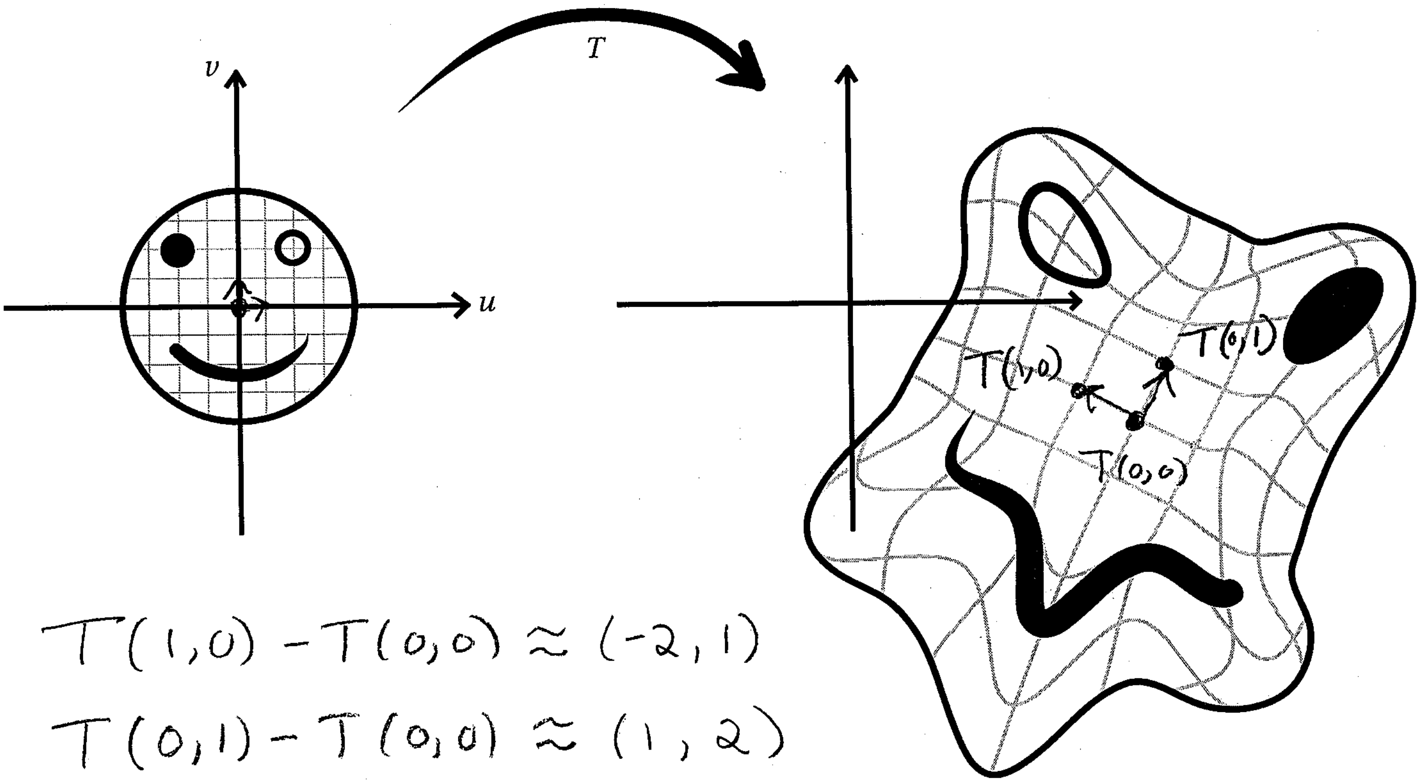
$D = \{0 \leq u \leq 1,\ 0 \leq v \leq 2\pi\}$

$\mathbf{r}(u, v) = \langle (1 - u)\cos v,\ v,\ (1 - u)\sin v \rangle$

(b) The integral $\iint_C y\, dS$ is: negative zero **positive**

Extra Credit: Consider the transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ which distorts the plane as shown below. Draw in $T(0,0)$ on the right-hand part of the picture and compute the Jacobian matrix of T at $(0,0)$, taking it as given that the entries of the matrix are integers and that the grid at left is made of unit squares. Be sure to explain your answer.

Note: If you need a makeshift ruler, you can tear off part of the upper right corner of this sheet.



$T(1,0) - T(0,0) \approx (-2,1)$

$T(0,1) - T(0,0) \approx (1,2)$

T is close to linear near $(0,0)$ as adjacent squares go to near squares. So

Jacobian matrix is $\approx \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$.