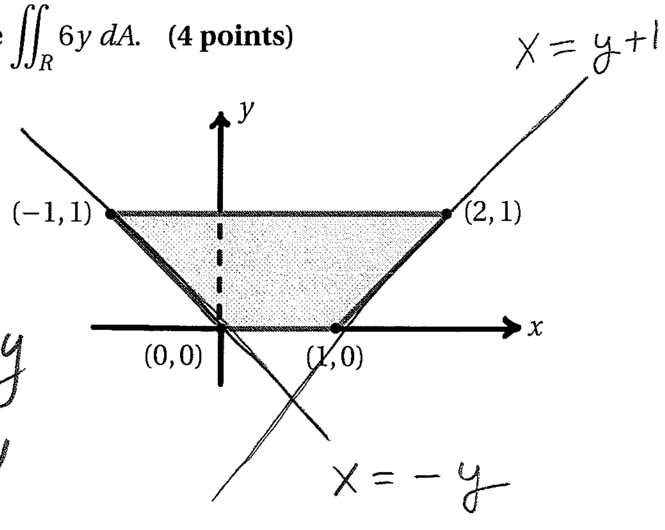
1. Let R be the region of integration pictured below right. Evaluate $\iint_R 6y\,dA$. **(4 points)**



$$\int_0^1\int_{-y}^{y+1} 6y\,dx\,dy$$

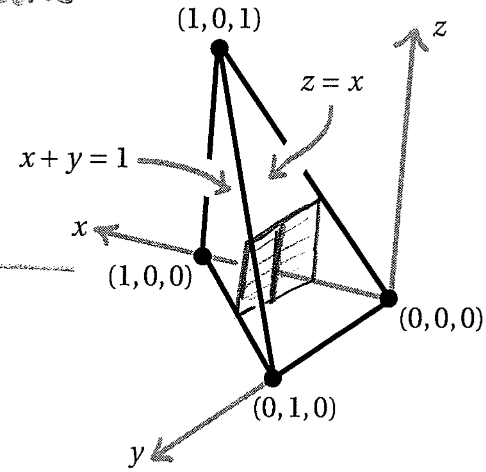
$$=\int_0^1 6yx\Big|_{x=-y}^{y+1} dy = \int_0^1 6y(2y+1)\,dy$$

$$=\int_0^1 12y^2+6y\,dy = 4y^3+3y^2\Big|_{y=0}^{y=1}$$

$$=7$$

$$\boxed{\iint_R 6y\,dA = 7}$$

2. Set up, but DO NOT EVALUATE, a triple integral that computes the volume of the tetrahedron shown at right. **(5 points)** Two common correct ans:

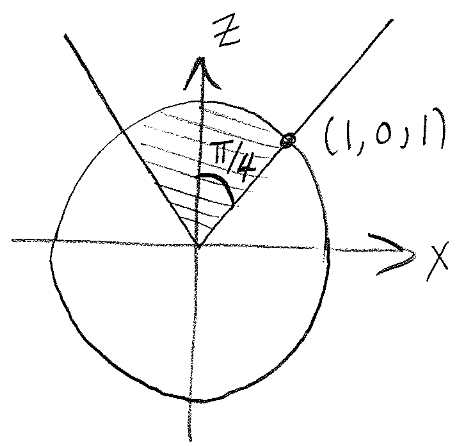


(a) $$\int_0^1\int_0^{1-x}\int_0^{x} 1 \quad dz\,dy\,dx$$

(b) $$\int_0^1\int_0^{1-y}\int_0^{x} 1 \quad dz\,dx\,dy$$

3. Set up, but DO NOT EVALUATE, a triple integral that computes the volume of the region that lies inside the sphere $x^2+y^2+z^2=2$ and above the cone $z=\sqrt{x^2+y^2}$. **(5 points)**

Cross section:



3D:

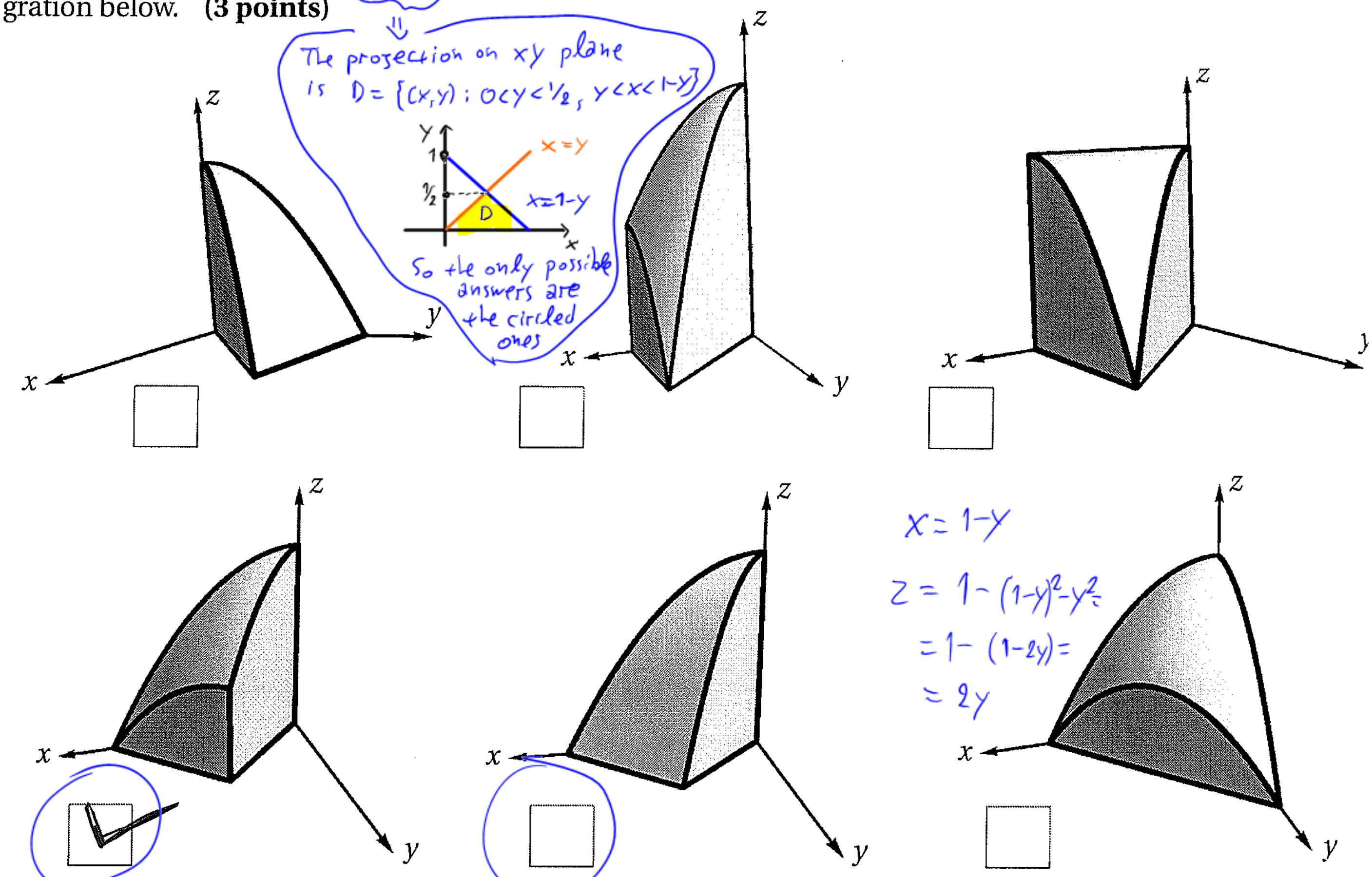
Using spherical coordinates.

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sqrt{2}} \underbrace{\rho^2\sin\phi\; d\rho\, d\phi\, d\theta}_{=1\,dV}$$

4. Consider the triple integral $\int_0^{1/2}\int_y^{1-y}\int_0^{1-x^2-y^2} f(x,y,z)\;dz\,dx\,dy$. Mark the corresponding region of integration below. **(3 points)**

The projection on xy plane is $D=\{(x,y): 0<y<1/2,\ y<x<1-y\}$

So the only possible answers are the circled ones

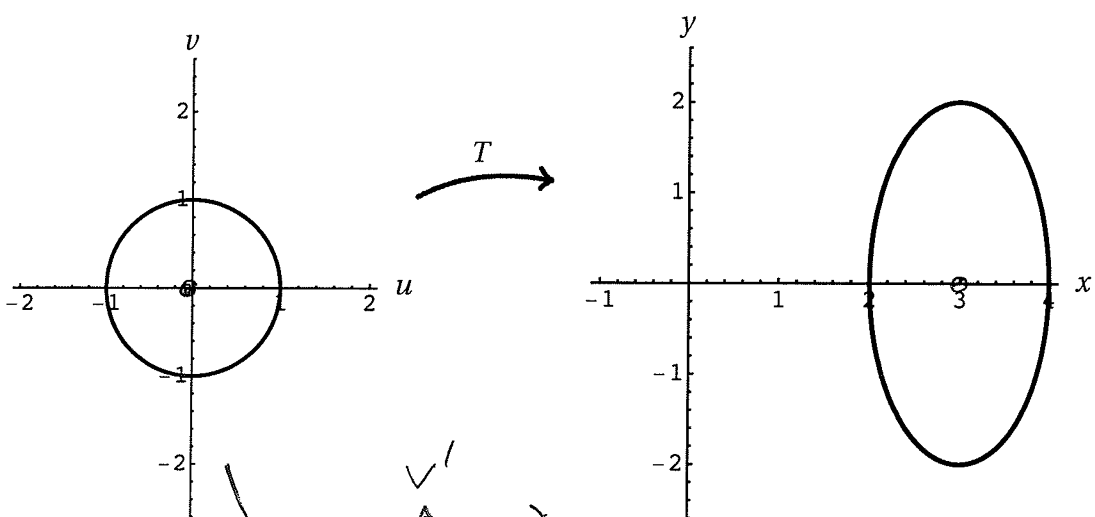


$x=1-y$

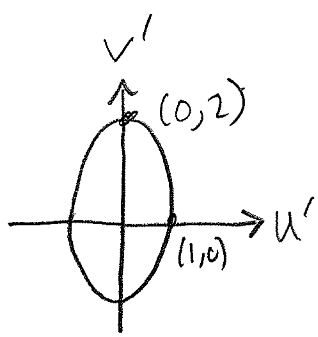
$z=1-(1-y)^2-y^2$
$=1-(1-2y)=$
$=2y$

But when $x=1-y$, then $z=1-(1-y)^2-y^2=1-(1-y-y)(1-y+y)=2y$ while in the right circled one $z=0$ when $x=1-y$. So it is not the right circled graph.

5. Find a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ which takes the unit circle to the ellipse given by $(x-3)^2+\frac{y^2}{4}=1$ as shown. **(3 points)**



$(u,v)\mapsto(u,2v)$

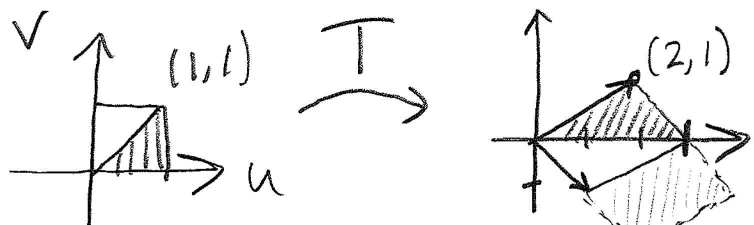


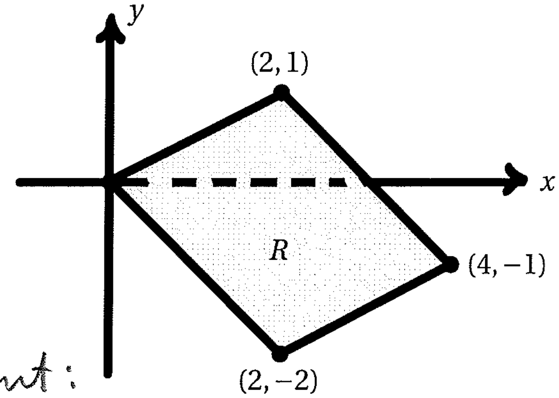
$(u',v')\mapsto(u'+3,v')$

$T(u,v) = (u+3\ ,\ 2v\ )$

6. Let R be the region in the xy-plane depicted below right. Let $T(u,v) = (2u+v, u-v)$.

(a) Find a rectangle S in the uv-plane whose image under T (that is, the collection of points $T(u,v)$ for all choices of (u,v) in S) is exactly R. **(3 points)**





$T(1,0) = (2,1)$
$T(0,1) = (1,-1)$

So $0 \le u \le 1$.
and to get image down to $(2,-2)$ want:

ANSWER: $S = \left\{(u,v) \,\middle|\, \boxed{0} \le u \le \boxed{1},\ \boxed{0} \le v \le \boxed{2}\right\}$.

(b) Set up, but DO NOT EVALUATE, the integral $\iint_R \cos(x)\, dA$ as an integral in the (u,v)-coordinates. If you can't do part (a), leave the limits of integration blank. **(5 points)**

$$\iint_R \cos x \, dA = \int_0^2 \int_0^1 \cos(2u+v)\, |\det J| \, du\, dv$$

$$= \int_0^2 \int_0^1 3\cos(2u+v)\, du\, dv$$

$$J = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\det J = -2-1 = -3$$

$$\boxed{\iint_R \cos(x)\, dA = \int_0^2 \int_0^1 3\cos(2u+v)\, du\, dv}$$

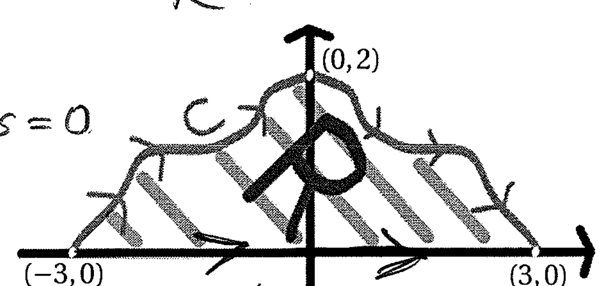
7. Let $\mathbf{F}(x,y) = \langle x^2, x^2\cos(y)\rangle$. Then $\iint_R \left[\frac{\partial}{\partial x}(x^2\cos(y)) - \frac{\partial}{\partial y}(x^2)\right] dA = 0$ where R is the region shown below. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the pictured curve that goes from $(-3,0)$ to $(3,0)$ via $(0,2)$. **(3 points)**

By Green, we have $\int_{C-C'} \vec{F}\cdot d\vec{r} = \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = 0$

Thus $\int_C \vec{F}\cdot d\vec{r} = \int_{C'} \vec{F}\cdot d\vec{r}$

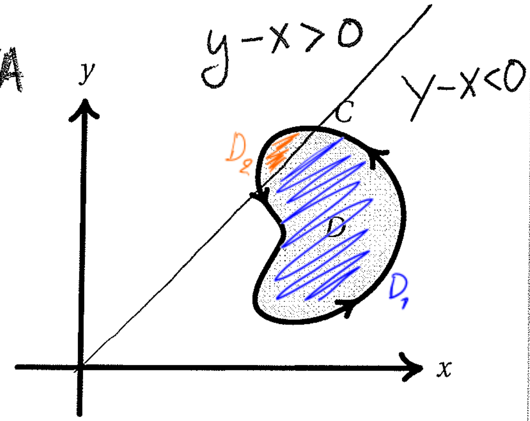
$= \int_{C'} x^2 dx + \underbrace{\int_{C'} x^2\cos y\, dy}_{\text{always } = 0}$

$= \int_{-3}^{3} x^2 dx = \left.\frac{x^3}{3}\right|_{-3}^{3} = \boxed{18}$



C' = bottom of R.

8. Consider the region D in the plane bounded by the curve C as shown at right. For each part, circle the best answer. **(1 point each)**



(a) For $\mathbf{F}(x,y) = \langle \overset{P}{x+1}, \overset{Q}{y^2} \rangle$, the integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is

$= \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint_R 0\, dA = 0$

negative **zero** positive

(b) The integral $\int_C (\overset{P}{-y}\,dx + \overset{Q}{2}\,dy)$ is

$= \iint_R \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint_R 1\, dA = \text{Area}(R)$

negative zero **positive**

(c) The integral $\iint_D (y-x)\, dA$ is

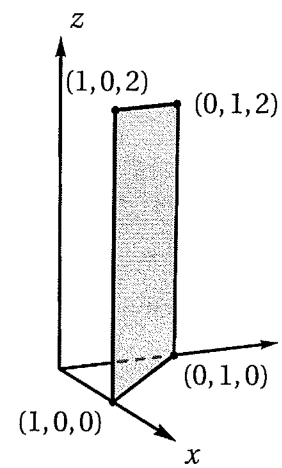
Split D into D_1 and D_2.

Then $I = \iint_D (y-x)\,dA = \underbrace{\iint_{D_1} (y-x)\,dA}_{<0} + \underbrace{\iint_{D_2} (y-x)\,dA}_{>0}$ and $\left|\iint_{D_1}\right| > \left|\iint_{D_2}\right|$, so I has to be negative.

negative zero positive

9. For each surface S below, give a parameterization $\mathbf{r}\colon D \to S$. Be sure to explicitly specify the domain D and call your parameters u and v.

(a) The rectangle in $\mathbb{R}^3$ with vertices $(1,0,0), (0,1,0), (1,0,2), (0,1,2)$. **(3 points)**

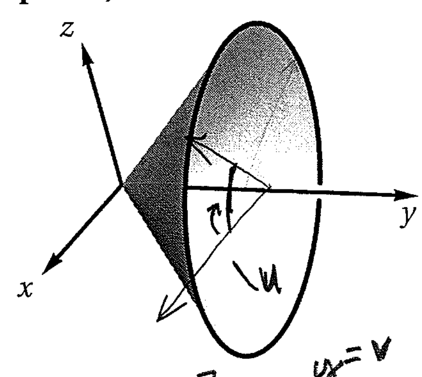


Can use coordinates x and z as parameters, call them u and v. Since this rect. is contained in the line $x+y=1$, get $y = 1-x = 1-u$

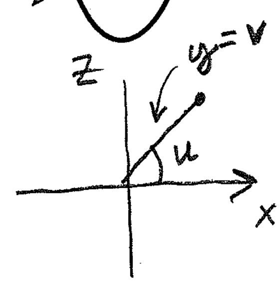
$D = \{\, 0 \le u \le 1$ and $0 \le v \le 2 \,\}$

$\mathbf{r}(u,v) = \langle\, u,\ 1-u,\ v \,\rangle$

(b) The portion of cone $y = \sqrt{x^2+z^2}$ for $0 \le y \le 1$ which is shown at right. **(4 points)**



Can use $v = y$ and the angle u shown as the parameters. Also, the radius of the circle with y fixed is just y, and hence:



$D = \{\, 0 \le u \le 2\pi$ and $0 \le v \le 1 \,\}$

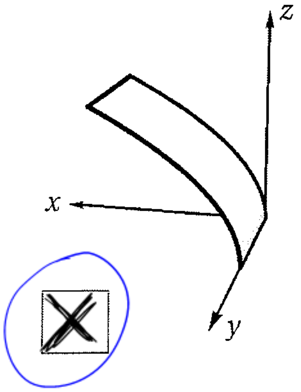
$\mathbf{r}(u,v) = \langle\, v\cos u,\ v,\ v\sin u \,\rangle$

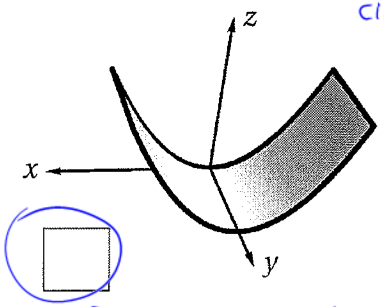
10. Consider the surface S parameterized by $\mathbf{r}(u,v) = (v^2,\ u,\ v)$ for $0 \le u \le 1$ and $0 \le v \le 1$.

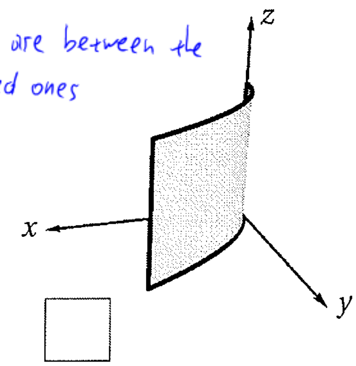
(a) Mark the correct picture of S below. **(2 points)**

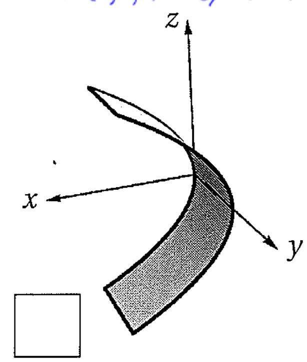
Along $x=0$ we get $v^2=0 \Rightarrow v=0$ so the pnts on the surface will be $(0,u,0) \in$ (y-axis)

So we are between the circled ones









But the x-coordinate of the surface is $v^2 \ge 0$, so it can't be the right circled one.

(b) Evaluate the integral $\iint_S z\, dA$. **(6 points)**

$$\iint_S z\, dA = \int_0^1\int_0^1 v\, |\vec{r}_u \times \vec{r}_v|\, du\, dv = \int_0^1\int_0^1 v\sqrt{1+4v^2}\, du\, dv$$

$$= \int_0^1 v\underbrace{\sqrt{1+4v^2}}_{w}\, dv = \int_1^5 \frac{1}{8}\sqrt{w}\, dw$$

$$dw = 8v\, dv$$

$$= \frac{1}{12} w^{3/2}\Big|_{w=1}^{w=5} = \frac{1}{12}\left(5^{3/2}-1\right)$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 2v & 0 & 1 \end{vmatrix} = (1, 0, -2v)$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{1+4v^2}$$

$$\boxed{\iint_S z\, dA = \frac{1}{12}\left(5^{3/2}-1\right)}$$

11. Consider the solid described as follows using cylindrical coordinates: E is the region inside the paraboloid $z = 1 - r^2$ and where $0 \le \theta \le \pi$ and $z \ge 0$. Choose one double integral and one triple integral below that compute the volume of E. **(1 point each)**

☐ $\displaystyle\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{\sqrt{1-x^2-y^2}} 1\, dz\, dx\, dy$

☑ $\displaystyle\int_0^1\int_0^{\sqrt{1-z}} 2\sqrt{1-z-y^2}\, dy\, dz$

☑ $\displaystyle\int_0^1\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_0^{\sqrt{1-z-x^2}} 1\, dy\, dx\, dz$

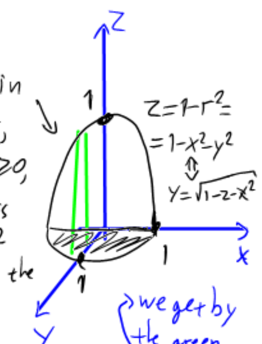
☐ $\displaystyle\int_0^1\int_0^{\sqrt{1-z^2}} 2\sqrt{1-y^2-z^2}\, dy\, dz$

☐ $\displaystyle\int_0^1\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_0^{1-\sqrt{x^2+y^2}} 1\, dz\, dx\, dy$

☐ $\displaystyle\int_0^1\int_0^{1-z} 2\sqrt{(1-z)^2-y^2}\, dy\, dz$

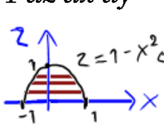
we get the double int. by equality with the triple one

cut in half, i.e. $y \ge 0$, bc θ is the angle of pnts of E with the x-axis.



we get by the green lines

On the xy-plane, $y=0 \Rightarrow z=1-x^2$



$z=1-x^2 \Leftrightarrow x = \pm\sqrt{1-z}$

so the two first int. $\int_0^1\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_0^{\cdots} 1\, dy\, dx\, dz$. and the 3rd one.