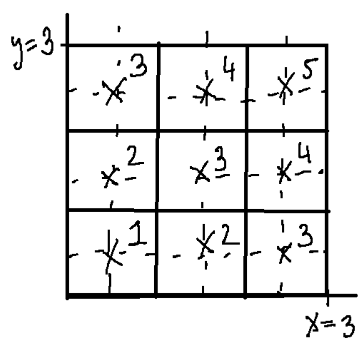
MIDTERM 3 FAQ

(1) **Question:** How do I estimate an integral of the form $\iint_R f\, dA$?
Answer: For a question of this type, the region will usually be simple enough for you to break it up into pieces and estimate via something like a Riemann Sum. Say you want to estimate an integral $\int\int_R f(x,y)\, dA$, where R is the square:

$$R = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 3, 0 \leq y \leq 3\}.$$

You can break this region into nine squares with area 1. Obtain a test value the function takes on each square, say by picking the center value. Alternatively, try to obtain an estimate of the average value obtained by f on each square. The information to be able to do this will be given to you in some form, for instance via a contour map or via a taable. In the figure below, we have obtained the value $f(x, y)$ of the given function f for the points marked by crosses. The number next to the crosses is the value of f at that site.

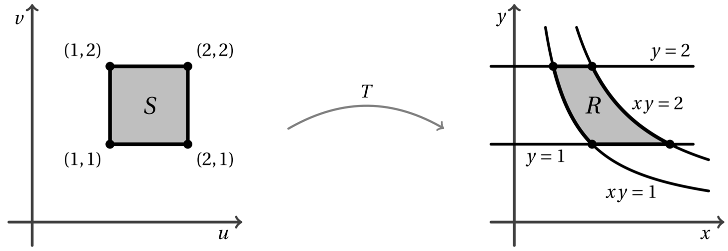


One then multiplies each value by the area of the corresponding square and adds all of these up. This is an estimate of the integral.

(2) **Question:** How do I determine what transformation will take one region to another?
Answer: Boundaries are mapped onto boundaries by the transformations we use here. So by guessing what transformation gets the boundary you have to the boundary you want, you will be almost done! Most of the transformations we use in this course will be to change variables on integrals, so see Section 15.9 of the textbook for examples.

(3) **Question:** How can I tell whether a transformation is a linear transformation or a non-linear transformation?
Answer: A transformation $T(u, v) = (g(u, v), h(u, v))$ is linear precisely when $g(u, v)$ and $h(u, v)$ are both linear in u and v. That is, it is linear when it is of the form $T(u, v) = (au + bv, cu + dv)$ for some constants $a, b, c, d \in \mathbb{R}$. Otherwise, T is non-linear.

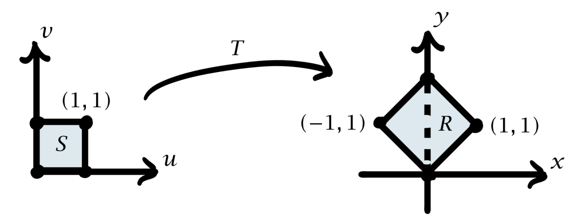
One can also tell if a transformation is non-linear based on where it sends certain regions. For instance, we know that a linear transformation sends straight lines to straight lines, so if you encounter a transformation T that sends some straight line to a curve that is not a line, then T cannot be a linear transformation. An example of such a scenario would be a transformation T that sends the square region S to the region R, as pictured below. Two of the edges of S must be mapped to curved edges in R, so T cannot be a linear transformation.



A linear transformation must also send the origin to the origin, so if the image of a region containing the origin does not contain the origin, then the transformation is not linear.

(4) **Question:** When determining what linear transformation will send one given region to another given region, is there more than one correct choice of linear transformation?
Answer: There can be more than one linear transformation that sends one region to another, depending on the restrictions that have been placed on the transformation.

Let us look at an example that illustrates such a scenario. Suppose we were required to construct a linear transformation T that would send the region S to the region R, as pictured below.



Recall that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is determined by where it sends any pair of non-parallel, non-zero vectors. In particular, our linear transformation T can be determined by where it sends the vectors $e_1 = \langle 1, 0\rangle$ and $e_2 = \langle 0, 1\rangle$. In order to transform the region S to the region R, we must send e_1 and e_2 to the vectors $f_1 = \langle 1, 1\rangle$ and $f_2 = \langle -1, 1\rangle$, but we can choose which vector to send to which.

- We could choose $T(e_1) = f_1$ and $T(e_2) = f_2$. This would give us the linear transformation
$$T(u, v) = uT(1, 0) + vT(0, 1) = u\langle 1, 1\rangle + v\langle -1, 1\rangle = \langle u - v, u + v\rangle.$$
- Alternatively, we could choose $T(e_1) = f_2$ and $T(e_2) = f_1$. This would give us the linear transformation
$$T(u, v) = uT(1, 0) + vT(0, 1) = u\langle -1, 1\rangle + v\langle 1, 1\rangle = \langle v - u, u + v\rangle.$$

Notice that both transformations send S to R, but each transformation sends the edges of S to different edges of R.[1]

If we had placed requirements on where the linear transformation sent specific edges, then our choice of linear transformation would have been limited. For example, if we had required that the linear transformation T send the edge of S given by the vector $\langle 1, 0\rangle$ to the edge of R given by the vector $\langle 1, 1\rangle$, then the only choice of linear transformation would have been $T(u, v) = \langle u - v, u + v\rangle$.

(5) **Question:** How do I rewrite an integral of the form $\iint_R f\, dA$ using a change of coordinates?
Answer: Say you have a change of coordinates:
$$x = g(u, v), \qquad y = h(u, v),$$
That is, you have a transformation $T(u, v) = (g(u, v), h(u, v))$ that takes some domain D to the domain R (the goal is usually for the domain D to be simpler). We transform the given integral,

[1] A key difference between these two transformations is that the first has a positive Jacobian and the second has a negative Jacobian. Transformations with positive Jacobians are referred to as "orientation preserving" and those with negative Jacobians are called "orientation reversing".

which is over R, to an integral over D. The integral becomes:

$$\iint_D f(T(u,v))\,|J(u,v)|\,du\,dv.$$

Let's look at each ingredient here.

First, the function f has been rewritten in terms of the u, v variables. This is done by replacing x and y with $g(u,v)$ and $h(u,v)$ respectively. Equivalently, we've plugged $T(u,v)$ into f to obtain $f(T(u,v))$.

Second, we have included the Jacobian $|J(u,v)|$. This amounts to taking the absolute value of the determinant:

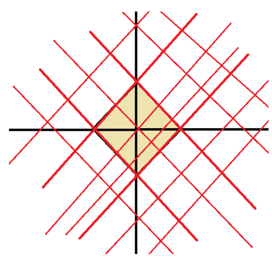
$$J(u,v) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

It's very important that you take the absolute value, else the result could be wrong. For example, the transformation $T(u,v) = (v,u)$ that exchanges the variables has $J(u,v) = -1$, and if you don't include the absolute value this would change the result by a sign.

Finally, the integral is now an integral over D, so you need to set up the new bounds using this region. Let's see an example. Say we want to integrate $\iint_R x\,dA$ where R is the region bounded by the lines $x+y=1$, $x-y=1$, $x+y=-1$ and $x-y=-1$. Define $u = x+y$, and $v = x-y$. Solving for x and y yields:

$$T(u,v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right).$$

The function $f(x,y) = x$ expressed as a function of (u,v) is $f(T(u,v)) = \frac{u+v}{2}$. If you set up and calculate the Jacobian, you get $J(u,v) = \frac{1}{2}$. The region R is highlighted in the following diagram:



The (u,v) coordinates are in red, the (x,y) coordinates in black. The bounding lines can be seen to correspond to the cases $u=1$, $u=-1$, $v=1$, and $v=-1$ respectively. So the transformation T takes the square $D = [-1,1]\times[-1,1]$ in the u,v-plane to the region R in the x,y-plane. So, the integral can be rewritten as:

$$\int_{-1}^{1}\int_{-1}^{1} \underbrace{\frac{u+v}{2}}_{f(T(u,v))}\ \underbrace{\frac{1}{2}}_{|J(u,v)|}\ du\,dv.$$

(6) **Question:** How can I use a double integral to compute the area of a region? How can I use a triple integral to compute the volume of a region?
Answer: To compute the area of a region R, you set up the integral $\iint_R dA$. That is, you do the double integral of the function $f(x,y) = 1$. Similarly, to compute the volume of a solid, you set up the integral $\iiint_E dA$. You can pick whichever coordinates are most conducive towards solving the problem, provided you remember to include the Jacobian. For example, suppose I want to calculate the area of the disk R, given by $\{(x,y) : x^2 + y^2 \leq 25\}$. Noting the rotational symmetry of this region, let's use polar coordinates. The area element is given by $dA = r\,dr\,d\theta$. The integral is then $\int_0^{2\pi}\int_0^5 r\,dr\,d\theta$.

(7) **Question:** How do I compute a triple integral using spherical coordinates?
Answer: Spherical coordinates are often used when surfaces such as cones and spheres form the boundary of the region of integration. The spherical coordinates are

$$x = \rho \cos\theta \sin\phi,$$
$$y = \rho \sin\theta \sin\phi,$$
$$z = \rho \cos\phi,$$

where ρ is the distance from the origin to the point (x, y, z), the angle θ is the angle between the point (x, y) and the x-axis, and ϕ is the angle between the z-axis and the line segment connecting the origin to the point (x, y, z). Note that $\rho \geq 0$ and $0 \leq \phi \leq \pi$. The general formula for a triple integral in spherical coordinates is the following:

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\phi)}^{h_2(\phi)} \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\rho^2 \sin\phi \, d\rho\, d\theta\, d\phi,$$

where

$$E = \{(\rho, \theta, \phi) \,|\, g_1(\theta, \phi) \leq \rho \leq g_2(\theta, \phi),\ h_1(\phi) \leq \theta \leq h_2(\phi),\ \alpha \leq \phi \leq \beta\}.$$

Take note that after we change the variables inside the integral, we need to multiply the quantity inside the integral by $\rho^2 \sin\phi$. Furthermore, the order of integration may be different, in which case the bounds would change too. To determine the bounds of integration, we must analyze the region of integration with spherical coordinates in mind. That is, for example, to determine the bounds on ρ in an integral in the order above, one might fix the angles ϕ and θ and see what the smallest and largest values of ρ are by looking at a ray emanating from the origin that makes the angles θ and ϕ in spherical coordinates. These values will be the functions $g_1(\theta, \phi)$ and $g_2(\theta, \phi)$ of θ and ϕ in the bounds above. See Section 15.8 of the textbook for further examples of integrating using spherical coordinates.

(8) **Question:** In general, how do you choose the order of integration for triple integrals?
Answer: It helps to pick the outside variable to be the one where the corresponding slices of the solid region are the easiest to visualize and/or describe by equations. Another thing to do is think of the outside two variables you pick as a double integral you are trying to set up, so you want the projection to the plane consisting of those two variables to be simple. Also, it's ok to just pick an order at random and then switch to a different one if things seem to be getting too messy. Sometimes, without actually trying to setup the integral, it can be hard to know in advance what is the best order.

(9) **Question:** How do I know how many parameters I need to use to parametrize a surface?
Answer: A surface is a two-dimensional object, so you need to use two variables. Similarly, to parametrize something three-dimensional, you need to use three variables, and to parametrize a curve, which is one-dimensional, you need only one parameter.

(10) **Question:** How do I compute a surface integral using a given surface parametrization?
Answer: There are two types of surface integrals discussed in this course: integrals of a function with respect to the surface area element dS, and integrals of vector fields through the surface.

Suppose you want to compute the first case: the integral $I = \iint_S f(x, y, z)\, dS$ for a function f given the parametrization of S as $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ where x, y, z are all functions from $\mathbb{R}^2$ to $\mathbb{R}$. You will need to find the partial derivatives $\mathbf{r}_u$ and $\mathbf{r}_v$, where $\mathbf{r}_u = \langle x_u, y_u, z_u\rangle$ and $\mathbf{r}_v = \langle x_v, y_v, z_v\rangle$. Then you will need to take the cross product $\mathbf{r}_u \times \mathbf{r}_v$. Finally, you will need the magnitude of that cross product $|\mathbf{r}_u \times \mathbf{r}_v|$. Then the integral is given by:

$$I = \iint_D f(x(u, v), y(u, v), z(u, v))|\mathbf{r}_u \times \mathbf{r}_v|\, dA,$$

where D is the region giving the domain of the parametrization. Thus, you need to use the region D to set up in the u, v-plane to set up the bounds of the double integral.

Now suppose you want to compute the second case: the integral $\iint_S \mathbf{F}(x, y, z) \cdot d\mathbf{S}$ of the vector field $\mathbf{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ across S using the parametrization $\mathbf{r}$ of S, then the surface integral is:

$$\iint_D \mathbf{F}(x(u, v), y(u, v), z(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA,$$

where again D is the region giving the domain of the parametrization. Notably, pay attention to the fact that you don't use the magnitude $|\mathbf{r}_u \times \mathbf{r}_v|$ of the cross product here, but instead take the dot product of the vector field evaluated at the parametrization with the cross product $\mathbf{r}_u \times \mathbf{r}_v$.

(11) **Question:** How do you estimate a surface integral $\iint_S f(x, y, z)\, dS$?
Answer: Think of it as a Riemann sum:
(a) chop the surface into small patches;
(b) for each patch, you multiply the value of the function $f(x, y, z)$ evaluated at some point in the patch, say the center, times the surface area of the patch;
(c) add up the results of (2).

Another approach is to think in terms of the average value of the function over the surface. Using the fact that:

$$\text{Average}(f) = \frac{1}{\iint_S dS} \iint_S f\, dS,$$

one can estmate $\iint_S f\, dS$ by estimating the average value of f and the surface area and multiplying these since the above implies:

$$\iint_S f\, dS = \text{Average}(f)\text{SurfaceArea}(S).$$

(12) **Question:** How do we tell when a surface integral is positive, negative, or zero?
Answer: When considering an integral of the form $\iint_R f\, dS$, look at whether f is mostly positive or mostly negative on the surface R. If the function is mostly positive on the surface R, then we would expect the answer to be positive. If the function is mostly negative, we expect the answer to be negative. As an example, let's say the surface R is the unit sphere $\{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ and the function is $f(x, y, z) = z^2$. This function is non negative everywhere on the unit sphere, so the integral must also be non negative.

Sometimes the integral is zero, and this usually hinges on some symmetry or some cancellation. For instance, look at the integral $\iint_R z\, dS$, where again R is the unit sphere and $f(x, y, z) = z$. Note that the function we are integrating is positive on the north side of the sphere, and negative on the south side. Furthermore, the function is odd in the z-variable since $f(x, y, -z) = -z = -f(x, y, z)$, and the sphere is symmetric with respect to reflection across the xy-plane. Intuitively, each z from a point on the north hemisphere "cancels" with a z from a point on the south hemisphere. Hence, the integral is zero.