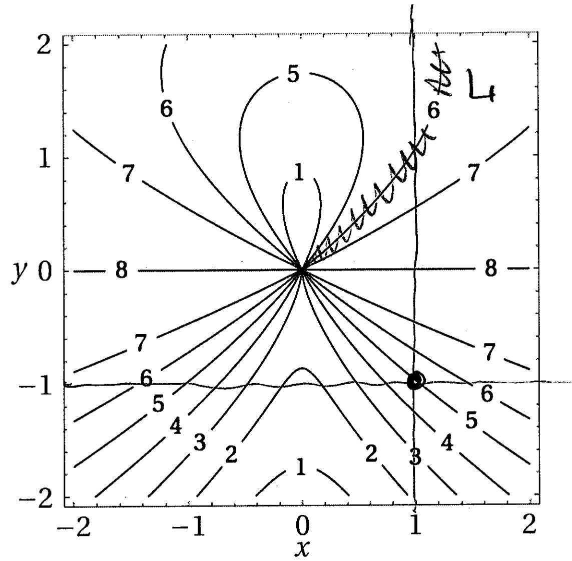
1. Consider the function $f(x,y)$ whose contour map is shown at right, where the value of f on each level curve is indicated by the number along it. For each part, give the answer that is **most consistent** with the given data. **For (a) and (b) be sure to explain your reasoning in the space provided.** If the limit does not exist, write "DNE" in the answer box.



(a) Determine $\lim_{x\to 0} f(x,0)$. **(2 points)**

As $f(x,0) = 8$ for all $x \neq 0$,
the limit exists and is 8

$\lim_{x\to 0} f(x,0) =$ 8

(a) Determine $\lim_{(x,y)\to(0,0)} f(x,y)$. **(2 points)**

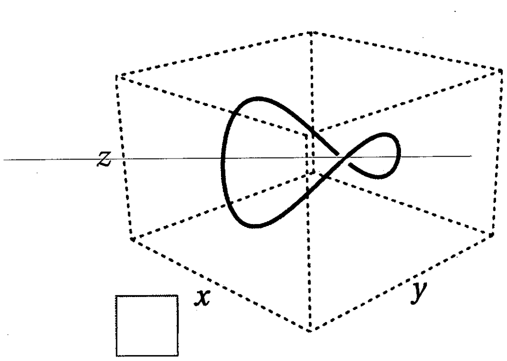
If instead we approach $(0,0)$ along the contour line L indicated, $f(x,y) = 6$.
Consequently, the limit does not exist

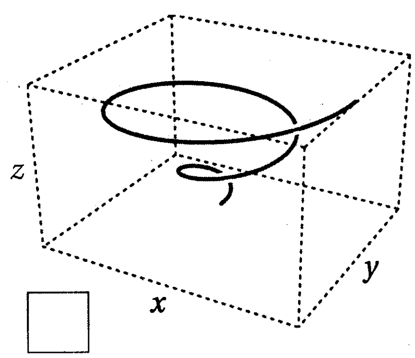
$\lim_{(x,y)\to(0,0)} f(x,y) =$ DNE

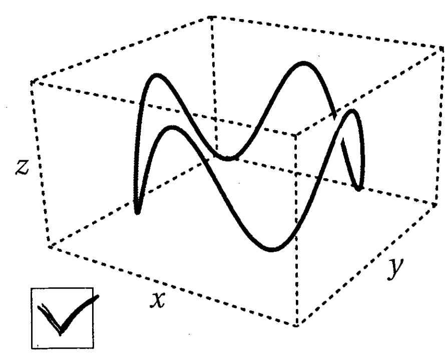
(c) Determine $\lim_{(x,y)\to(1,-1)} f(x,y)$. **(1 point)**

$\lim_{(x,y)\to(1,-1)} f(x,y) =$ 5

2. Mark the box next to the curve that is parameterized by $\mathbf{r}(t) = \langle 2\cos(t),\ 2\sin(t),\ \cos(4t)\rangle$ for $0 \le t \le 2\pi$. **(2 points)**







3. Suppose $f\colon \mathbb{R}^2 \to \mathbb{R}$ has the table of values and partial derivatives shown at right. For $x(s,t) = s + 2t$ and $y(s,t) = s^2 - t$, let $F(s,t) = f\big(x(s,t), y(s,t)\big)$ be their composition with f. Compute $\frac{\partial F}{\partial \cancel{t}\, s}(2,1)$. **(4 points)**

(x,y)	$f(x,y)$	$\frac{\partial f}{\partial x}$	$\frac{\partial f}{\partial y}$
$(2,1)$	0	7	6
$(2,-1)$	-12	7	-1
$(3,3)$	19	-8	5
$(4,3)$	7	3	2

For $(s,t) = (2,1)$, we have $x = 4$ and $y = 3$.

So

$$\frac{\partial F}{\partial s}(2,1) = \frac{\partial f}{\partial x}(4,3)\,\frac{\partial x}{\partial s}(2,1) + \frac{\partial f}{\partial y}(4,3)\,\frac{\partial y}{\partial s}(2,1)$$

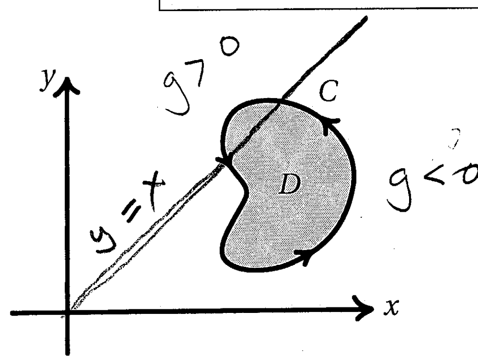
$$= 3 \cdot 1 + 2 \cdot 4$$

$$\frac{\partial x}{\partial s} = 1$$

$$\frac{\partial y}{\partial s} = 2s \;\; @ \; (2,1) = 4$$

$\frac{\partial F}{\partial \cancel{t}\, s}(2,1) =$ 11

4. Consider the region D in the plane bounded by the curve C as shown at right. For each part, circle the best answer. **(1 point each)**



P Q

(a) For $\mathbf{F}(x,y) = \langle x+1,\ y^2 \rangle$, the integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ is

negative	zero	positive

P Q

(b) The integral $\int_C (y\,dx + 2\,dy)$ is

negative	zero	positive

g

(c) The integral $\iint_D (y - x)\, dA$ is

negative	zero	positive

Scratch Space

a) $Q_x - P_y = 0 - 0$ so Green says $\int_C = 0$

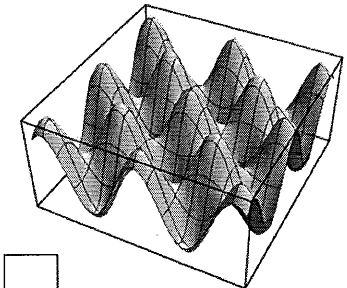
b) $Q_x - P_y = 0 - 1$ So Green says

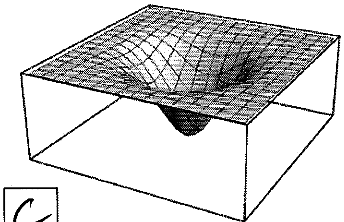
$\int_C y\,dx + 2\,dy = \iint_D -1\, dA = -\text{Area}(D) < 0.$

c) the integrand is negative on the bulk of D so integral is negative.

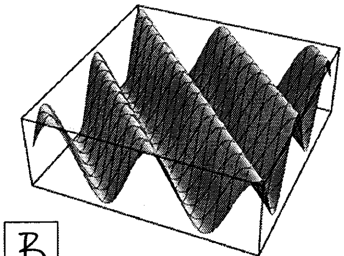
5. For each function label its graph from among the options below: **(1 point each)**

(A) $x^2 - y^2$ (B) $\cos(x+y)$ (C) $1 - e^{-(x^2+y^2)}$

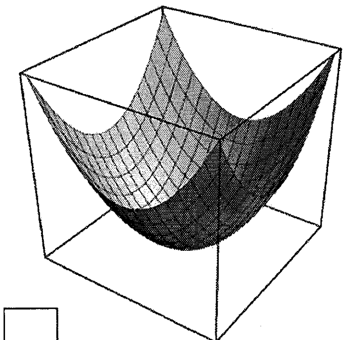


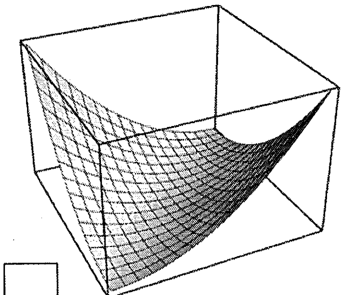


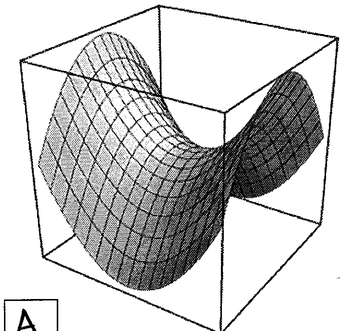
C



B







A

6. A rectangular metallic plate R is placed in the plane with vertices at $(-2,-1)$, $(-2,1)$, $(2,-1)$, and $(2,1)$. The density (in g/cm^2) of the plate, $\rho(x,y)$, at various points is shown in the table, where x and y are measured in cm. Circle the best estimate for the mass of the plate. **(2 points)**

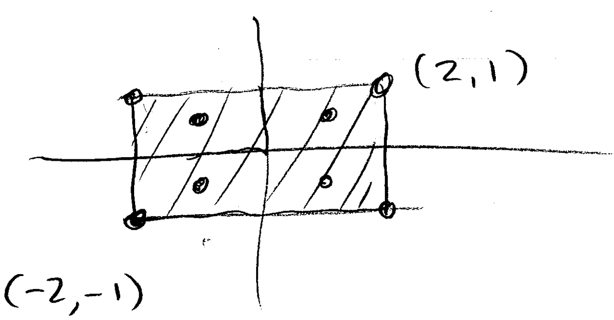
$\rho(x,y)$		$x = -1$	$x = 1$
y	$1/2$	4	7
	$-1/2$	1	3

Mass of $R \approx$

0	4	15	30	46	60	78

grams.

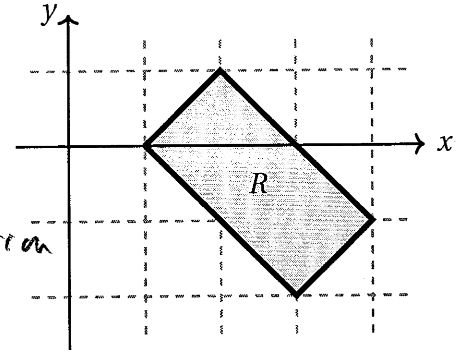
Scratch Space



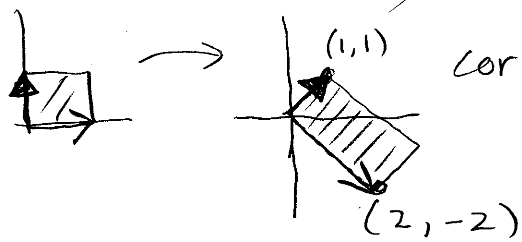
Each subrectangle centered at a sample point has area 2 so mass is $\approx$

$2 \cdot (4 + 7 + 1 + 3) = 30$

7. Let R be the rectangle whose vertices are $(1,0)$, $(2,1)$, $(3,-2)$, and $(4,-1)$ shown at the right.



(a) Find a transformation $T(u,v)$ from the uv–plane to the xy–plane with $T(S) = R$, where $S = \{(u,v) \mid 0 \le u \le 1, 0 \le v \le 1\}$ is the unit square. **(4 points)** Lets do a linear transformation



cor. to $A = \begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$

and then shift x by +1.

Also correct $(u+2v+1,\ u-2v)$

$T(u,v) = (2u+v+1 \quad , -2u+v \quad)$

(b) Use your answer in (a) to fill in the integrand below to evaluate $\iint_R \cos(x)\, dA$ by a change of coordinates. **(2 points)**

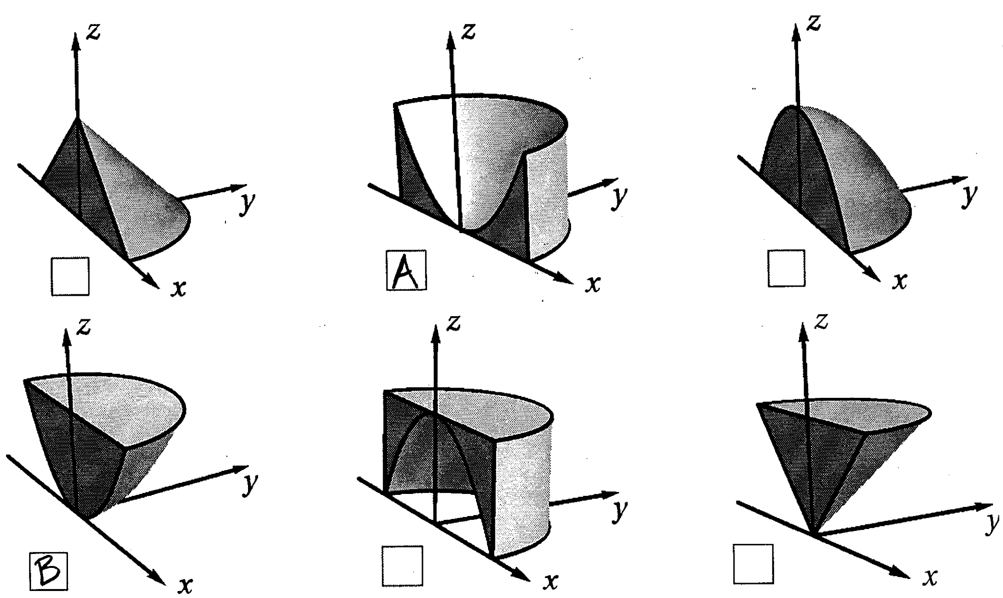
Jacobian matrix is $\begin{pmatrix} 2 & 1 \\ -2 & 1 \end{pmatrix}$ with $|\det| = 4$.

$$\iint_R \cos(x)\, dA = \int_0^1 \int_0^1 \cos(2u+v+1)\ 4 \ du\, dv$$

8. For each of the integrals below, label the solid corresponding to the region of integration. **(2 points each)**

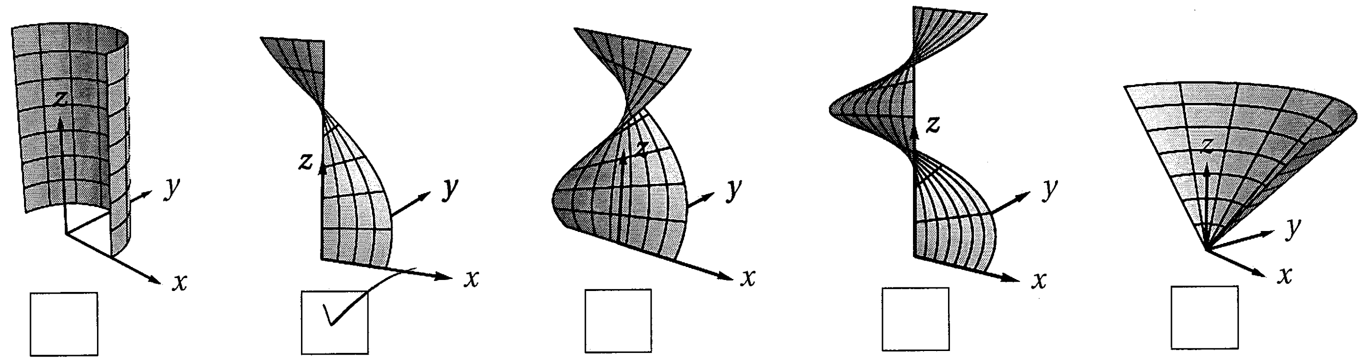
(A) $\displaystyle\int_0^{\pi} \int_0^1 \int_0^{r^2} f(r,\theta,z)\, r\, dz\, dr\, d\theta$

(B) $\displaystyle\int_0^{\pi} \int_0^1 \int_0^{\sqrt{z}} f(r,\theta,z)\, r\, dr\, dz\, d\theta$



9. Let S be the surface in $\mathbb{R}^3$ parameterized by $\mathbf{r}(u,v) = \langle u\cos(v),\ u\sin(v),\ v\rangle$ for $0 \le u \le 1$ and $0 \le v \le \pi$.

(a) Check the box below the correct picture of S. **(2 points)**



(b) Fill in the integrand below so that the integral computes $\iint_S y\,dS$. **(4 points)**

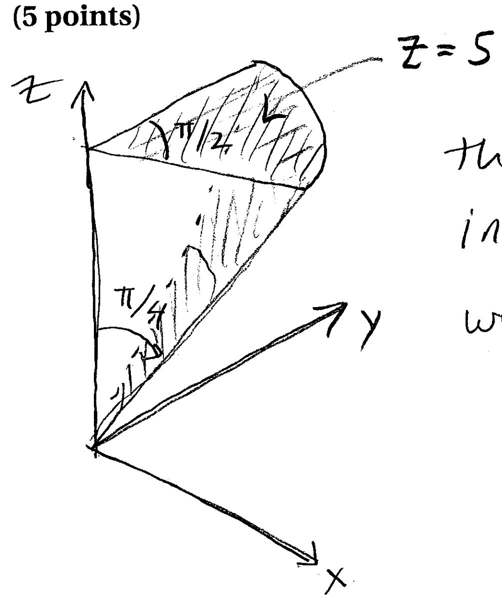
$$r_u \times r_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = (\sin v, -\cos v, u)$$

so $dS = |r_u \times r_v|\,du\,dv = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}$

So $\iint_S y\,dS = \iint (u\sin v)\sqrt{1+u^2}\,du\,dv$

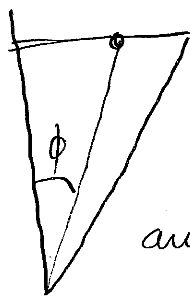
$$\boxed{\iint_S y\,dS = \int_0^\pi \int_0^1 (u\sin v)\ \sqrt{1+u^2}\ du\,dv}$$

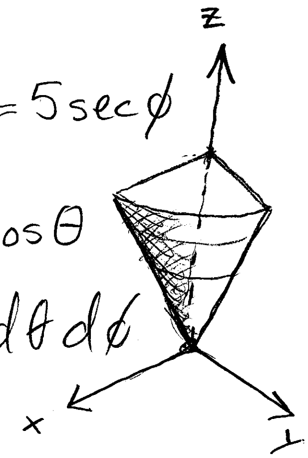
10. Let R be the region in the positive octant that lies above the cone $x^2 + y^2 = z^2$ and below the plane $z = 5$. Suppose R is made of material of density $\rho(x, y, z) = x$. Fill in the limits and integrand so that the integral computes the mass of R using spherical coordinates. Be sure to follow the provided order of integration. **(5 points)**



Here $0 \leq \phi \leq \pi/4$ to stay above the cone and $0 \leq \theta \leq \pi/2$ to stay inside the pos. octant. For ρ we have $z = \rho \cos\phi$, so

$$0 \leq \rho \leq 5/\cos\phi = 5\sec\phi$$





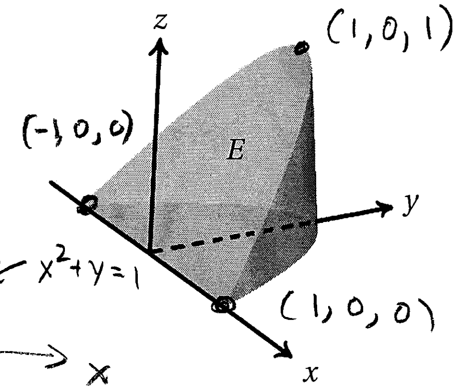
As $\rho = x = \rho\sin\phi\cos\theta$ and $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

the integrand is $\rho^3 \sin^2\phi \cos\theta$.

$$\text{mass} = \int_0^{\pi/4} \int_0^{\pi/2} \int_0^{5\sec\phi} \rho^3 \sin^2\phi \cos\theta \; d\rho \, d\theta \, d\phi$$

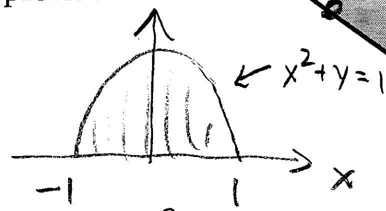
Scratch Space

11. Consider the region E shown at right, which is bounded by the xy-plane, the plane $z - y = 0$ and the surface $x^2 + y = 1$.

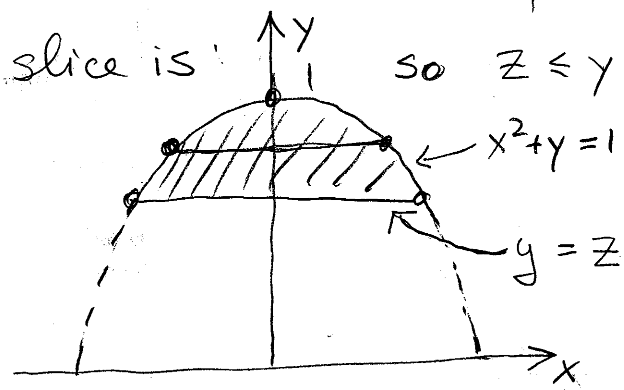


(a) Fill in the limits and integrand of the triple integral below so that it computes the volume of E. Be sure to follow the provided order of integration. **(4 points)**

Viewed from above E looks like



Have $0 \le z \le 1$ from picture, and for fixed z our slice is:



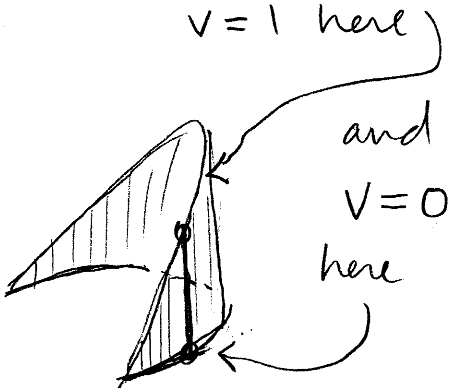
so $z \le y \le 1$. For fixed y, x goes from $-\sqrt{1-y}$ to $+\sqrt{1-y}$.

$$\text{Vol}(E) = \int_0^1 \int_z^1 \int_{-\sqrt{1-y}}^{\sqrt{1+y}} 1 \; dx\, dy\, dz$$

(b) Let S be the curved portion of the boundary of E where $x^2 + y = 1$. Parameterize S by $\mathbf{r}\colon D \to S$, where the domain D is a rectangle. **(4 points)**

Params: $u = x$

v = vertical with $v = 1$ here



and $v = 0$ here

So $x = u$

$y = 1 - x^2 = 1 - u^2$

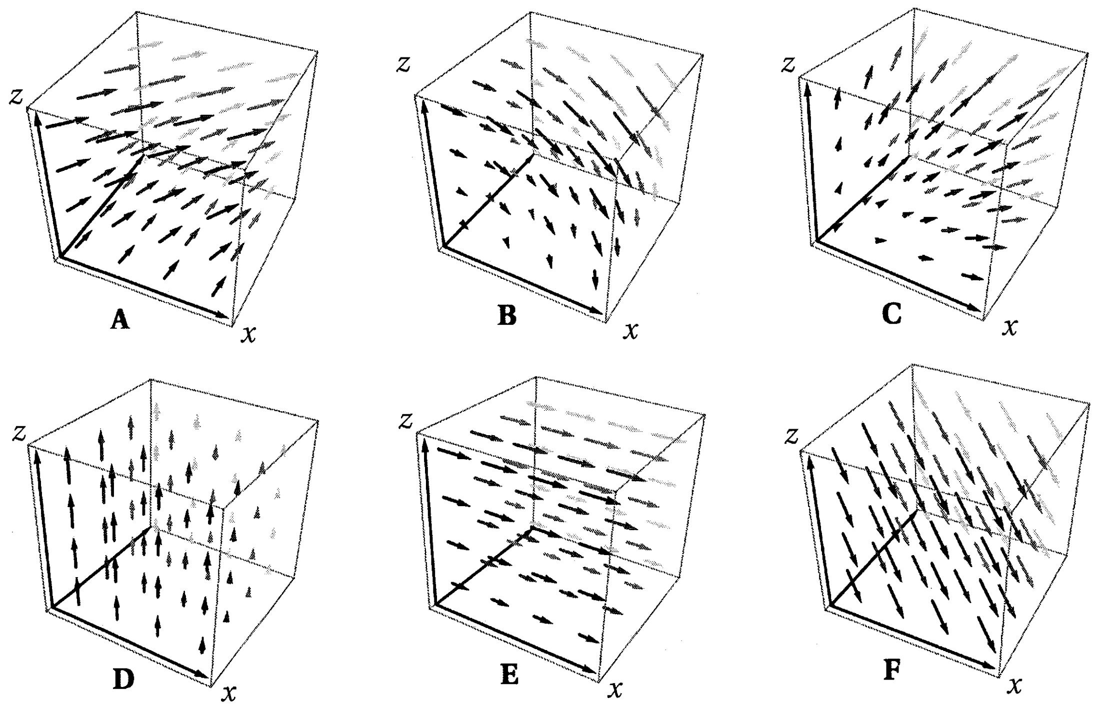
$z = (1-v)\cdot 0 + v\cdot(1-u^2)$

$= v(1-u^2)$

$D = \{ -1 \le u \le 1 \text{ and } 0 \le v \le 1 \}$

$\mathbf{r}(u, v) = \langle\, u \,,\; 1 - u^2 \,,\; v(1-u^2) \,\rangle$

12. Here are plots of six vector fields on the box where $0 \le x \le 1$, $0 \le y \le 1$, and $0 \le z \le 1$. For each part, circle the best answer. **(1 point each)**



(a) The vector field given by $\langle z,\ 1,\ 0\rangle$ is: **A B C D E F**

(b) Exactly one of these vector fields has nonzero divergence. It is: **A B C D E F**

For this vector field, the divergence is generally: negative positive

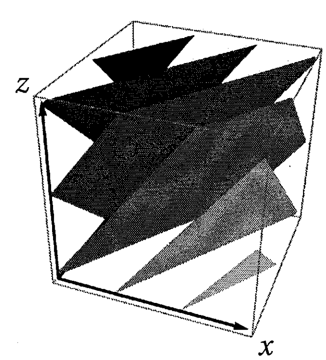
(c) The vector field **D** is conservative: true false

(d) Exactly one of the vector fields is constant, that is, independent of position. It is: **A B C D E F**

(e) The vector field curl **B** is constant. The value of curl **B** is:

i $-$**i** **j** $-$**j** **k** $-$**k** **0**

(f) The vector field that is the gradient of a function f whose level sets are shown at right is: **A B C D E F**

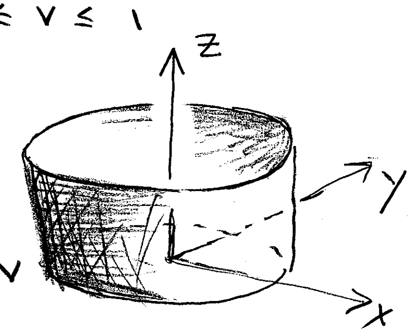


13. Let R be the solid region in $\mathbb{R}^3$ bounded by the cylinder $x^2+y^2=1$ and the planes $z=0$ and $z=1$. Divide the boundary ∂R into three parts: the bottom B where $z=0$, the top T where $z=1$, and the curved surface S where $x^2+y^2=1$. Orient all of these surfaces by the normal vectors that point out of R. Consider the vector field $\mathbf{F}=\langle 2x,\ 0,\ 1-z\rangle$.

(a) Compute the flux of $\mathbf{F}$ through each of S, T, and B. (Hint: $\int_0^{2\pi}\sin^2 t\,dt=\int_0^{2\pi}\cos^2 t\,dt=\pi$.) **(6 points)**

$\underline{S}$: param by $\vec{r}(u,v)=\langle \cos u, \sin u, v\rangle$ $\quad 0\le u\le 2\pi$, $0\le v\le 1$

$$\vec{r}_u\times\vec{r}_v=\begin{vmatrix} i & j & k\\ -\sin u & \cos u & 0\\ 0 & 0 & 1\end{vmatrix}=\langle \cos u, \sin u, 0\rangle$$



$$\iint_S \vec{F}\cdot\vec{n}\,dS=\int_0^1\int_0^{2\pi}\langle 2\cos u, 0, 1-v\rangle\cdot\langle\cos u,\sin u,0\rangle\,du\,dv$$

$$=\int_0^1\int_0^{2\pi}2\cos^2 u\,du\,dv \overset{\text{hint}}{=} \int_0^1 2\pi\,dv=2\pi$$

$\underline{T}$: On T, $F=\langle 2x,0,0\rangle$ (as $z=1$) and $\vec{n}=\langle 0,0,1\rangle$, so $\vec{F}\cdot\vec{n}=0$ and hence $\iint_T \vec{F}\cdot\vec{n}\,dS=0$

$\underline{B}$: On B, have $\vec{n}=\langle 0,0,-1\rangle$ and so $\iint_B \vec{F}\cdot\vec{n}\,dS$
$=\iint_B\langle 2x,0,1\rangle\cdot\langle 0,0,-1\rangle\,dA=\iint_B -1\,dA=-\text{Area}(B)=-\pi$

$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=$ 2π	$\iint_T \mathbf{F}\cdot\mathbf{n}\,dS=$ 0	$\iint_B \mathbf{F}\cdot\mathbf{n}\,dS=$ $-\pi$

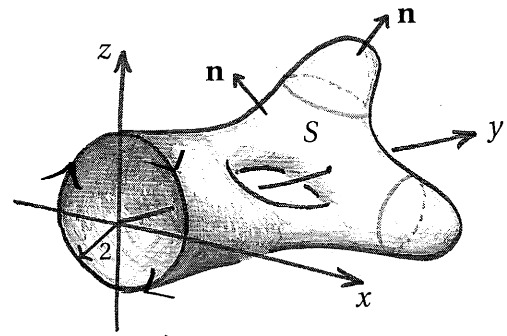
(b) Use the Divergence Theorem to check your computation for the flux through ∂R in (a–c). **(2 points)**

$\iint_{\partial R}\vec{F}\cdot\vec{n}\,dS=$ sum of ans in (a-c) $=2\pi+0-\pi=\pi$

Div thm says $=\iiint_R \operatorname{div}\vec{F}\,dV=\iiint_R 2-1\,dV=$
$\text{Vol}(R)=\pi$ as needed

$\iint_{\partial R}\mathbf{F}\cdot\mathbf{n}\,dS=$ π.

14. The surface S shown at right has boundary the circle C of radius 2 in the xz-plane.



(a) Consider the vector field $\mathbf{F} = \langle z, y, -x \rangle$ on $\mathbb{R}^3$. With respect to the normal vector field $\mathbf{n}$ indicated, compute the flux of $\operatorname{curl} \mathbf{F}$ through S. **(5 points)**

Use Stokes', so orient C as shown which can param by $\vec{r}(t) = (2\sin t, 0, 2\cos t)$ for $0 \le t \le 2\pi$.

Now $\int_{\partial S} \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \underbrace{\langle 2\cos t, 0, -2\sin t \rangle}_{\vec{F}(\vec{r}(t))} \cdot \underbrace{\langle 2\cos t, 0, -2\sin t \rangle}_{\vec{r}\,'(t)} dt$

$= \int_0^{2\pi} 4\cos^2 t + 4\sin^2 t \, dt$

$= \int_0^{2\pi} 4 \, dt = 8\pi$, and so Stokes' gives

$$\boxed{\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 8\pi}$$

(a) Consider the vector field $\mathbf{G} = \langle 3z + e^{\sin x}, e^{\cos y}, \sin(e^z) \rangle$. With $C = \partial S$ oriented **counter-clockwise** from our viewpoint, compute $\int_C \mathbf{G} \cdot d\mathbf{r}$. **(4 points)**

Let D be the disk in the xz plane with $\partial D = C$. Orient D by $\vec{m} = \langle 0, -1, 0 \rangle$ to match c-c orient. By Stokes

$\int_C \vec{G} \cdot d\vec{r} = \iint_D (\operatorname{curl} \vec{G}) \cdot \vec{m} \, dA = \iint_D -3 \, dA = -3 \operatorname{Area}(D)$

$= -3 \cdot 2^2 \pi$

as $\operatorname{curl} \vec{G} = \langle 0, 3, 0 \rangle$

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3z + e^{\sin x} & e^{\cos y} & \sin(e^z) \end{vmatrix} = \langle 0, 3, 0 \rangle$$

$$\boxed{\int_C \mathbf{G} \cdot d\mathbf{r} = -12\pi}$$

15. Consider the vector fields $\mathbf{A} = \langle 2x,\ z,\ y \rangle$ and $\mathbf{B} = \langle x,\ 0,\ y \rangle$ and $\mathbf{C} = \langle y,\ 0,\ z \rangle$. **(1 point each)**

(a) Circle the unique vector field that is conservative: $\boxed{\mathbf{A} \quad \mathbf{B} \quad \mathbf{C}}$

pot. fn $= x^2 + yz = f$

$f = 4$

(b) Suppose $\mathbf{F}$ is your answer in (a) and W is any curve starting at $(2, 0, -1)$ and ending at $(1, 1, 1)$.

$f = 2$

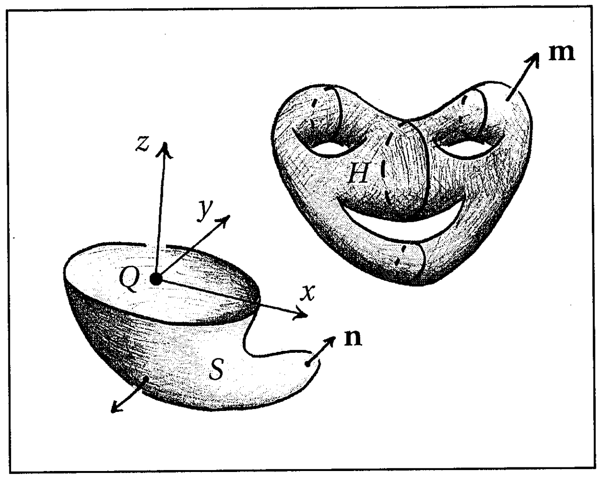
Circle the value of $\int_W \mathbf{F} \cdot d\mathbf{r}$: $\boxed{-5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5}$

$= f(1,1,1) - f(2,0,-1) = -2$

(c) Circle the unique vector field that is curl $\mathbf{G}$ for some vector field $\mathbf{G}$: $\boxed{\mathbf{A} \quad \mathbf{B} \quad \mathbf{C}}$

since div $\vec{C} = 0$

16. Let S and H be the surfaces at right; the boundary of S is the unit circle in the xy-plane, and H has no boundary. Suppose there is a **negative** charge Q placed at the origin and let $\mathbf{E}$ be the resulting electrical field. For each part, circle the correct answer. **(1 point each)**



(a) The flux $\iint_S \mathbf{E} \cdot \mathbf{n}\, dS$ is: $\boxed{\text{negative} \quad \text{zero} \quad \text{positive}}$

(b) The flux $\iint_H \mathbf{E} \cdot \mathbf{m}\, dS$ is: $\boxed{\text{negative} \quad \text{zero} \quad \text{positive}}$

(c) Consider the sphere S where $x^2 + y^2 + z^2 = 100$. The flux with respect to the outward normals $\mathbf{u}$ is:

$\boxed{\iint_S \mathbf{E} \cdot \mathbf{u}\, dS = Q/\varepsilon_0}$

Gauss's Law.

(d) The integral $\iint_H x^2 y^2 + z^2\, dS$ is: $\boxed{\text{negative} \quad \text{zero} \quad \text{positive}}$

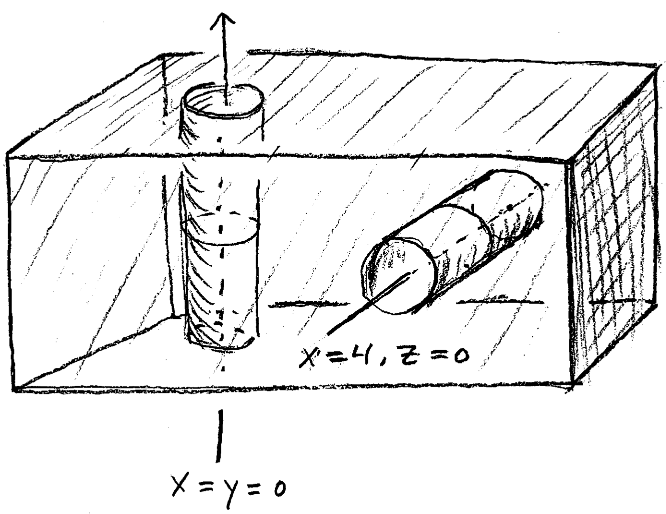
since integrand is > 0 everywhere.

Scratch Space

Extra Credit: (5 points)
Consider the solid region $E = \{-2 \le x \le 6$ and $-2 \le y \le 2$ and $-2 \le z \le 2$ and $x^2 + y^2 \ge 1$ and $(x-4)^2 + z^2 \ge 1\}$ inside $\mathbb{R}^3$.

(a) Draw an accurate picture of E.
(b) Give a vector field $\mathbf{F}$ on E where $\operatorname{curl}\mathbf{F} = 0$ but $\mathbf{F}$ is not conservative.
(c) Find a second vector field $\mathbf{G}$ on E where $\operatorname{curl}\mathbf{G} = 0$ and $\mathbf{G}$ is not conservative where $\mathbf{G} \neq a\mathbf{F} + \nabla h$ for all a in $\mathbb{R}$ and differentiable functions $h\colon E \to \mathbb{R}$.

a) E is a rectangular block with two radius one holes drilled out, one centered on the z-axis and the other around the line $x = 4, z = 0$.



b) For $(x,y) \neq (0,0)$ set

$$\vec{F} = \frac{1}{x^2+y^2}\langle -y, x, 0\rangle$$

This has $\operatorname{curl}\vec{F} = \vec{0}$ by the calc. on honors HW 4. $\vec{F}$ is defined on all of E and is not conservative there as if we orient the circle $C_0 = \{x^2+y^2 = 2, z = 0\}$ counter clockwise then

$$\int_{C_0} \vec{F}\cdot d\vec{r} = \int_{C_0} \vec{F}\cdot\vec{T}\, ds = \int_{C_0} \frac{1}{\sqrt{2}}\, ds = = 2\pi > 0$$

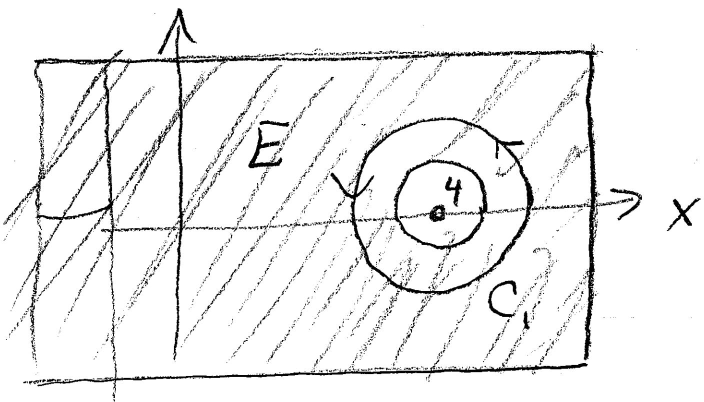
parallel

and so $\vec{F}$ is not path independent.

c) We rotate and shift $\vec{F}$ so that the z-axis becomes the core of the other tube: $\vec{G} = \frac{1}{(x-4)^2+z^2}\langle -z, 0, x-4\rangle$ for $(x,z) \neq (4,0)$

which is again defined on all of E. Since we moved everything by a rigid motion, will still have $\text{curl}\,\vec{G} = \vec{0}$ (or check directly).

Let C_1 be the circle $\{(x-4)^2 + z^2 = 2,\ y = 0\}$ oriented as shown. Again, we have $\int_{C_1} \vec{G}\cdot d\vec{r} = 2\pi$ so $\vec{G}$ is not conservative.



If $\vec{G}$ were $a\vec{F} + \nabla h$ where a is in $\mathbb{R}$ and $h: E \to \mathbb{R}$, then

$$\int_{C_1} \vec{G}\cdot dr = \underbrace{a\int_{C_1} \vec{F}\cdot d\vec{r}}_{} + \underbrace{\int_{C_1} \nabla h \cdot dr}_{0 \text{ by F.T.L.I}} = 0 \text{ a contradiction.}$$

$= 0$ since F is defined on the disc D_1 in $\mathbb{R}^3$ bounded by C_1 and so Stokes' says

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{D_1} \text{curl}\,\vec{F}\cdot\vec{n}\, dA = \int_{D_1} 0\, dA = 0.$$