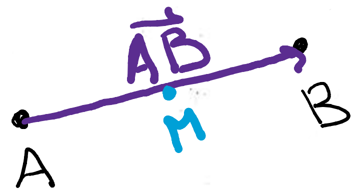
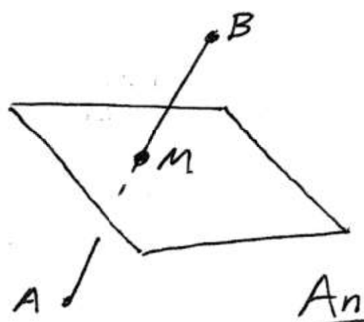
1. Consider the points $A = (2,0,1)$ and $B = (4,2,5)$ in $\mathbb{R}^3$.

(a) Find the point M which is halfway between A and B on the line segment L joining them. **(2 pts)**



$$M = A + \frac{1}{2}\vec{AB}$$
$$= (2,0,1) + \frac{1}{2}(2,2,4)$$
$$= (2,0,1) + (1,1,2)$$
$$= (3,1,3)$$

(b) Find the equation for the plane P consisting of all points that are equidistant from A and B. **(3 pts)**



Point on P: $M = (3,1,3)$

Normal to P: $\frac{1}{2}(\vec{w} - \vec{v}) = \frac{1}{2}((4,2,5) - (2,0,1))$

$$= \frac{1}{2}(2,2,4) = (1,1,2)$$

Ans: $1\cdot(x-3) + 1\cdot(y-1) + 2\cdot(z-3) = 0$

OR: $x + y + 2z = 10$

2. Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Compute the following limit, if it exists. **(4 pts)**

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

Along x-axis: $f(x,0) = \dfrac{x\cdot 0}{x^2+0^2} = 0$

Along $y = x$: $f(x,x) = \dfrac{x\cdot x}{x^2+x^2} = \dfrac{1}{2}$

So as f does not approach a consist. value as $(x,y)\to(0,0)$ the limit $\boxed{\text{does not exist}}$

(b) Where on $\mathbb{R}^2$ is the function f continuous? **(1 pts)**

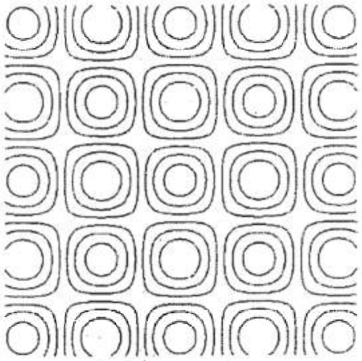
Everywhere except $(0,0)$.

Note: f is not cont. at $(0,0)$ b/c $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist

3. Match the following functions $\mathbb{R}^2 \to \mathbb{R}$ with their graphs and contour diagrams. Here each contour diagram consists of level sets $\{f(x,y) = c_i\}$ drawn for evenly spaced c_i. **(9 pts)**

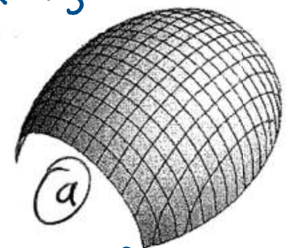
(a) $\sqrt{8 - 2x^2 - y^2}$ (b) $\cos x$ (c) xy

if $z = k$, $\cos x = k \Rightarrow$ x is constant

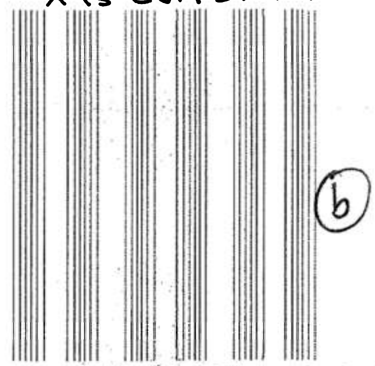


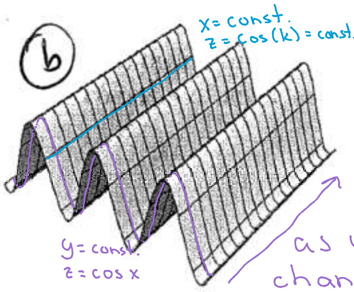
$z^2 = 8 - 2x^2 - y^2$

$2x^2 + y^2 + z^2 = 8$

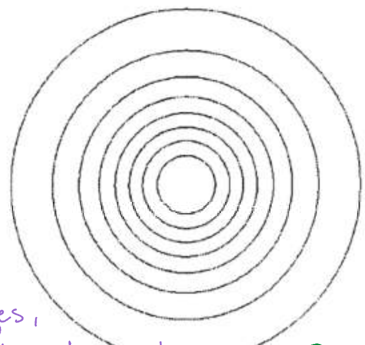


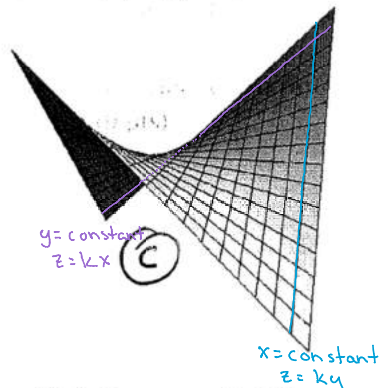
So, surface is part of an ellipsoid

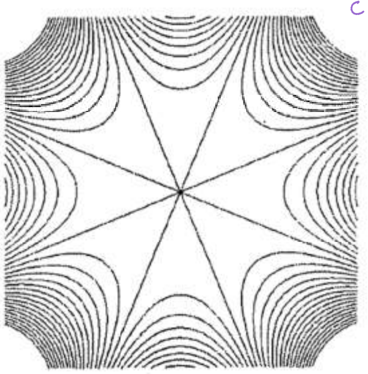


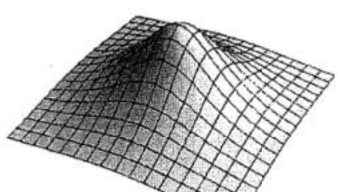


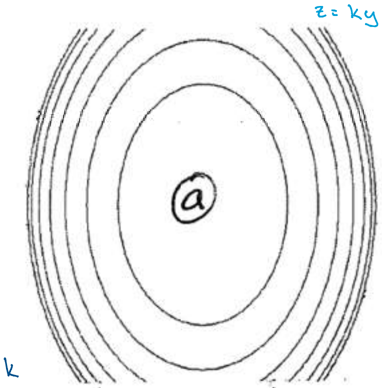
as y changes, function doesn't change







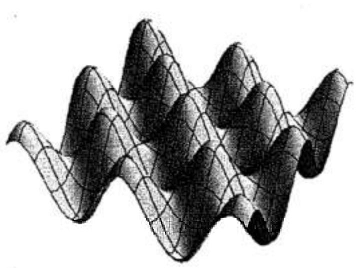


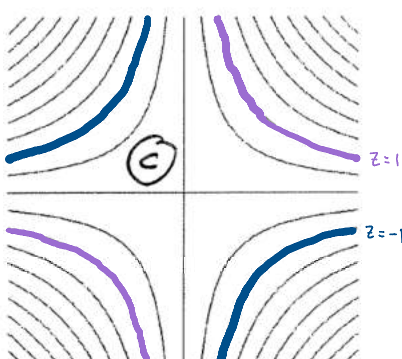


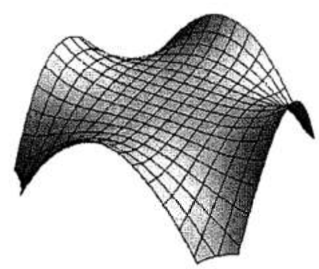
$z = k$

$k^2 = 8 - 2x^2 - y^2$

$2x^2 - y^2 = 8 - k^2$ ← ellipse







$xy = k \Rightarrow y = \frac{k}{x}$

z = constant

4. Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = xy$.

(a) Use Lagrange multipliers to find the global (absolute) max and min of f on the circle $\underbrace{x^2 + y^2 = 2}_{g(x,y)}$. **(6 pts)**

First

$$(y, x) = \nabla f = \lambda \nabla g = \lambda(2x, 2y).$$

Thus $y = 2\lambda x$ and $x = 2\lambda y$ and so $2\lambda = \frac{x}{y} = \frac{y}{x}$.

Therefore $x^2 = y^2$. Since also $x^2+y^2=2$, we get $2x^2 = 2$

$\Rightarrow x^2 = \pm 1$ and $y^2 = \pm 1$. So there are 4 critical pts

Point	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
Val of f	1	-1	-1	$+1$
Type	Max	Min	Min	Max

Note: Global min/max exist since C is closed and bounded.

(b) If they exist, find the global min and max of f on $D = \{x^2 + y^2 \leq 2\}$. **(2 pts)**

Need to check for critical pts. of f inside D. These occur when $\nabla f = \langle 0,0\rangle$

$\nabla f = \langle 2x, 2y\rangle = \langle 0,0\rangle$

$\Rightarrow (x,y) = (0,0)$.

Note: we don't need to classify this critical pt b/c EVT tells us max & min must both occur on D: either at a pt on boundary, or at a critical pt inside D.

$f(0,0) = 0$. Since $1 > 0$ & $-1 < 0$,

1 is global max & -1 is global min.

(c) For each critical point in the interior of D you found in part (b), classify it as a local min, local max, or saddle. **(2 pt)** $f_x = y$ $f_y = x$

$f_{xx} = 0$ $f_{yy} = 0$ $f_{xy} = f_{yx} = 1$

Check

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0$$ hence a saddle.

(d) If they exist, find the global min and max of f on $\mathbb{R}^2$. **(2 pts)**

Neither exist since $f \to +\infty$ along $y = x$ and $\to -\infty$ along $y = -x$.

(f is surface from 3 (c))

5. A function $f\colon \mathbb{R}^2 \to \mathbb{R}$ takes on the values shown in the table at right.

(a) Estimate the partials $f_x(1,1)$ and $f_y(1,1)$. **(2 pts)**

				x		
		0.2	0.6	1.0	1.4	1.8
	1.8	3.16	3.88	4.60	5.32	6.04
	1.4	2.68	3.24	3.80	4.36	4.92
y	1.0	2.20	2.60	3.00	3.40	3.80
	0.6	1.72	1.96	2.20	2.44	2.68
	0.2	1.24	1.32	1.40	1.48	1.56

$$f_x(1,1) \approx \frac{f(1.4,1)-f(1,1)}{0.4} = \frac{3.4-3.0}{0.4} = 1$$

$$f_y(1,1) \approx \frac{f(1,1.4)-f(1,1)}{0.4} = \frac{3.8-3.0}{0.4} = 2$$

(b) Use your answer in (a) to approximate $f(1.1, 1.2)$. **(2 pts)**

$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$$f(1.1, 1.2) \approx f(1,1) + f_x(1,1)\overbrace{(0.1)}^{\Delta x} + f_y(1,1)\overbrace{(0.2)}^{\Delta y}$$

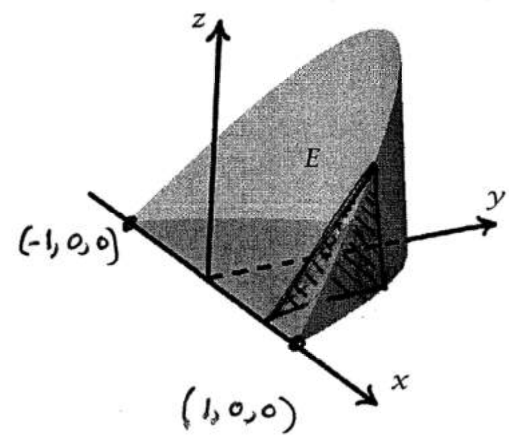
$$\approx 3 + 1.(0.1) + 2(0.2) = 3.5.$$

(c) Determine the sign of $f_{xy}(1,1)$: $\boxed{\text{positive}}$ negative zero **(1 pt)**

$$f_y(1.4,1) \approx \frac{f(1.4,1.4)-f(1.4,1)}{0.4} = \frac{4.36-3.40}{0.4} = \frac{0.96}{0.4} > 2.$$

Thus f_y increases as x increases $\Rightarrow \frac{\partial}{\partial x} f_y > 0$.

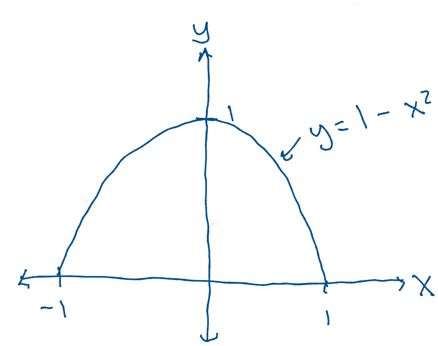
6. Consider the region E shown at right, which is bounded by the xy-plane, the plane $z - y = 0$ and the surface $x^2 + y = 1$. Complete setup, but do not evaluate, a triple integral that computes the volume of E. **(6 pts)**



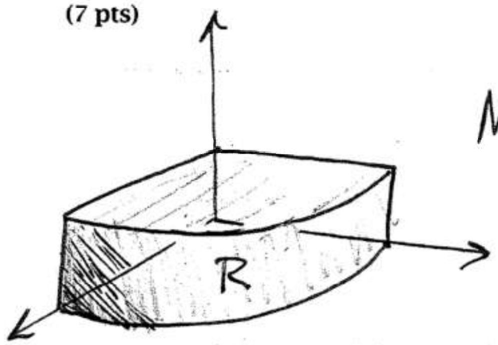
$$\int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{y} 1\, dz\, dy\, dx$$

Or

$$\int_{0}^{1}\int_{-\sqrt{1-y}}^{\sqrt{1-y}}\int_{0}^{y} 1\, dz\, dx\, dy$$



7. Consider the portion R of the cylinder $x^2 + y^2 \le 2$ which lies in the positive octant and below the plane $z = 1$. Compute the total mass of R when it is composed of material of density $\rho = e^{x^2+y^2}$. **(7 pts)**



$$\text{Mass} = \iiint_R \rho \, dV = \iiint_R e^{x^2+y^2} \, dV$$

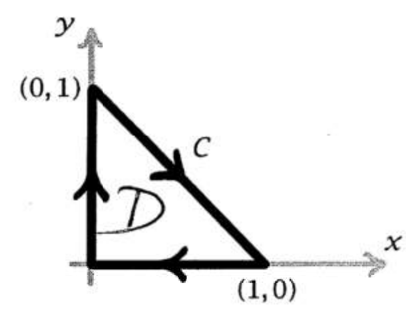
$$= \int_0^{\sqrt{2}} \int_0^1 \int_0^{\pi/2} e^{r^2} r \, d\theta \, dz \, dr = \int_0^{\sqrt{2}} \frac{\pi}{2} e^{r^2} r \, dr \qquad \begin{array}{l} u = r^2 \\ du = 2r\,dr \end{array}$$

using cylindrical coor.

$$= \int_0^2 \frac{\pi}{4} e^u \, du$$

$$= \frac{\pi}{4}(e^2 - 1).$$

8. For the curve C in $\mathbb{R}^2$ shown and the vector field $\mathbf{F} = (\overset{P}{\ln(\sin(x))}, \overset{Q}{\cos(\sin(y)) + x})$ evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ using the method of your choice. **(5 pts)**



Let's use Green's Thm:

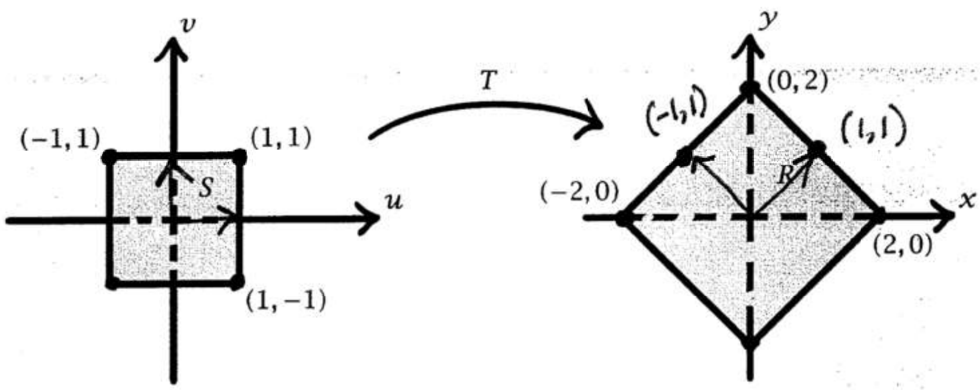
$$\int_C \vec{F} \cdot d\vec{r} = - \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$$

because
C is clockwise

$$= - \iint_D 1 - 0 \, dA$$

$$= - \text{Area}(\triangle) = -\frac{1}{2}$$

9. Let R be the region shown at right.



(a) Find a transformation T: $\mathbb{R}^2 \to \mathbb{R}^2$ taking $S = [-1,1] \times [-1,1]$ to R. **(4 pts)**

Want $T(1,0) = (1,1)$ $\quad T(0,1) = (-1,1)$. If

$T(u,v) = (au+bv, cu+dv)$ we can solve $T(1,0) = (a,c) = (1,1)$

$T(0,1) = (b,d) = (-1,1)$

to find $T(u,v) = (u-v, u+v)$.

Check: $T(1,1) = (1-1, 1+1) = (0,2)$

$T(1,-1) = (2,0)$

(b) Use your change of coordinates to evaluate $\iint_R y^2\, dA$ via an integral over S. **(6 pts)**

Emergency backup transformation: If you can't do (a), pretend you got the answer $T(u,v) = (uv, u+v)$ and do part (b) anyway.

$$\iint_R y^2\, dA = \iint_S (u+v)^2 \,|\det J|\, du\, dv$$

$$= \int_{-1}^{1}\int_{-1}^{1} 2(u^2 + 2uv + v^2)\, du\, dv$$

$$= 2\int_{-1}^{1} \frac{u^3}{3} + u^2 v + v^2 u \Big|_{u=-1}^{1} dv$$

$$= 2\int_{-1}^{1} \frac{2}{3} + 2v^2\, dv = 4\left(\frac{1}{3}v + \frac{v^3}{3}\right)\Big|_{v=-1}^{1}$$

$$= \frac{16}{3}.$$

$T = (g,h)$

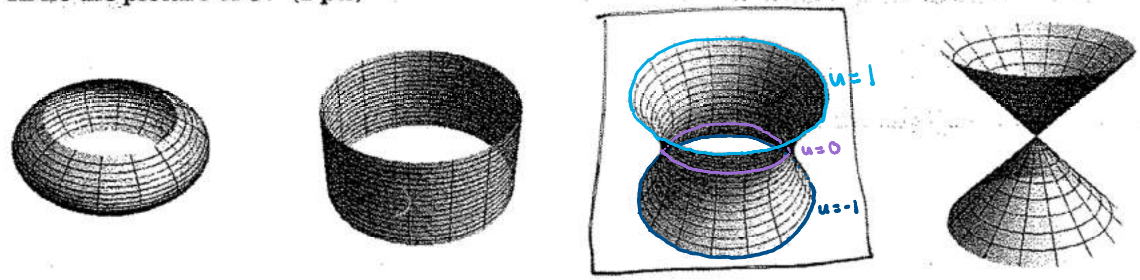
$$J = \begin{pmatrix} g_u & g_v \\ h_u & h_v \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$\det J = 2$

10. Consider the surface S which is parameterized by $\mathbf{r}(u,v) = (\sqrt{1+u^2}\cos v, \sqrt{1+u^2}\sin v, u)$ for $-1 \le u \le 1$ and $0 \le v \le 2\pi$. $u=-1: \vec{r}(-1,v) = (\sqrt{2}\cos v, \sqrt{2}\sin v, -1)$

$u=0: \vec{r}(0,v) = (\cos v, \sin v, 0)$

(a) Circle the picture of S. **(2 pts)** $u=1: \vec{r}(1,v) = (\sqrt{2}\cos v, \sqrt{2}\sin v, 1)$



(b) Completely setup, but do not evaluate, an integral that computes the surface area of S. **(6 pts)**

$$\text{Area} = \iint_S 1\, dA = \iint_D |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

$$= \int_0^{2\pi}\int_{-1}^{1} \sqrt{1+2u^2}\, du\, dv$$

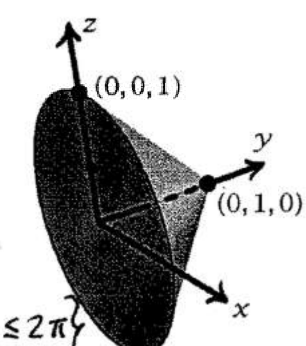
$$r_u \times r_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{u}{\sqrt{1+u^2}}\cos v & \frac{u}{\sqrt{1+u^2}}\sin v & 1 \\ -\sqrt{1+u^2}\sin v & \sqrt{1+u^2}\cos v & 0 \end{vmatrix}$$

$$= \left(\sqrt{1+u^2}\cos v, -\sqrt{1+u^2}\sin v, u\right)$$

$$\to |\vec{r}_u \times \vec{r}_v| = \sqrt{(1+u^2)\cos^2 v + (1+u^2)\sin^2 v + u^2}$$

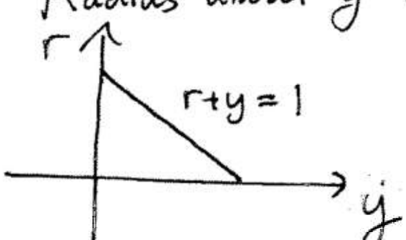
$$= \sqrt{1+2u^2}$$

11. For the cone S at right, give a parameterization $\mathbf{r}\colon D \to S$. Explicitly specify the domain D. **(5 pts)**



Params: $u = y$ $\quad v =$ angle about y-axis

Radius about y-axis is a fn of $y = u$



So take $D = \{0 \le u \le 1 \text{ and } 0 \le v \le 2\pi\}$

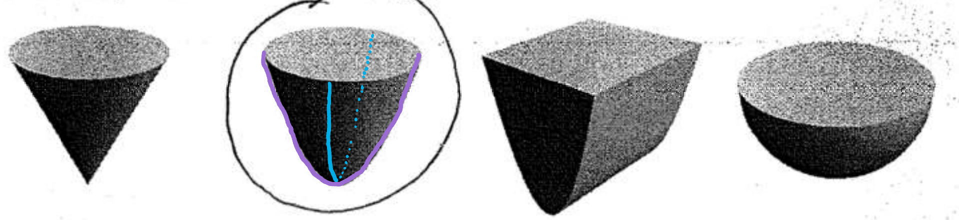
with:

$$\vec{r}(u,v) = \left((1-u)\cos v, u, (1-u)\sin v\right)$$

12. Consider the region R in $\mathbb{R}^3$ above the surface $x^2 + y^2 - z = 4$ and below the xy-plane. Also consider the vector field $\mathbf{F} = (0, 0, z)$.

$\left.\begin{array}{l} x=0: \ z = y^2 - 4 \\ y = 0: \ z = x^2 - 4 \end{array}\right\}$ parabolas

(a) Circle the picture of R below. (2 pts)



(b) Directly calculate the flux of $\mathbf{F}$ through the entire surface ∂R, with respect to the outward unit normals. **(10 pts)**

Now $\partial R = (T = \ )$ and $(S = \ )$. First, $\iint_T \vec{F} \cdot \vec{n}\, dA = \iint_T (0,0,0)\cdot(0,0,1)\, dA = \iint_T 0\, dA = 0$. Second, lets param. S by $\vec{r}(u,v) = (u, v, u^2+v^2-4)$ on $D = \{u^2+v^2 \le 4\}$. Then

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = (-2u, -2v, 1)$. As this points the wrong way we use $\vec{r}_v \times \vec{r}_u$ instead. Now Flux $= \iint_S (\vec{F}\cdot\vec{n})\, dA =$

$\iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_v \times \vec{r}_u)\, du\, dv = \iint_D (0, 0, u^2+v^2-4)\cdot(2u, 2v, -1)\, du\, dv$

$= \iint_D 4 - u^2 - v^2\, du\, dv = \int_0^2 \int_0^{2\pi} (4 - r^2) r\, d\theta\, dr = 2\pi \int_0^2 4r - r^3\, dr$

$= 2\pi \left(2r^2 - \frac{r^4}{4}\right)\Big|_{r=0}^{r=2} = 2\pi(8-4) = \boxed{8\pi}$

Hence $\iint_{\partial R} \vec{F}\cdot\vec{n}\, dA = \iint_T \vec{F}\cdot\vec{n}\, dA + \iint_S \vec{F}\cdot\vec{n}\, dA = 0 + 8\pi = \boxed{8\pi}$

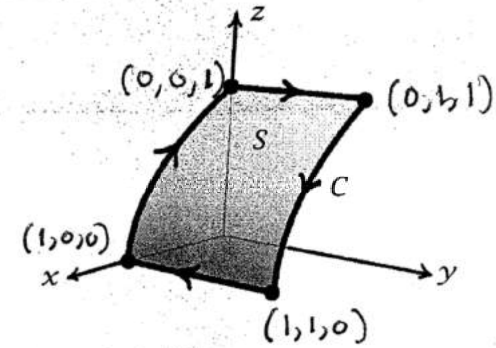
(c) Use the Divergence Theorem and your answer in (b) to compute the volume of R. **(3 pts)**

$\iint_{\partial R} \vec{F}\cdot\vec{n}\, dA = \iiint_R \operatorname{div}\vec{F}\, dV = \iiint 1\, dV = \text{Volume}.$

$\operatorname{div}\vec{F} = \frac{\partial 0}{\partial x} + \frac{\partial 0}{\partial y} + \frac{\partial z}{\partial z} = 1$

So: $\boxed{\text{Vol} = 8\pi.}$

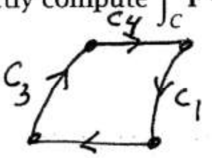
13. Let C be the curve shown at right, which is the boundary of the portion of the surface $x + z^2 = 1$ in the positive octant where additionally $y \leq 1$.



(a) Label the four corners of C with their (x, y, z)-coordinates. **(1 pt)**

(b) For $\mathbf{F} = (0, xyz, xyz)$, directly compute $\int_C \mathbf{F} \cdot d\mathbf{r}$. **(6 pts)**

Break C up into



Notice if <u>any</u> of x, y or z is 0, then $\vec{F} = \vec{0}$. Thus for any C_i except C_1, we have $\int_{C_i} \vec{F} \cdot d\vec{r} = \int_{C_i} \vec{0} \cdot d\vec{r} = 0$.

For C_1 we parameterize $-C_1$ via

$$\vec{r}(t) = (1 - t^2, 1, t) \quad \text{for } 0 \leq t \leq 1.$$

$$\text{So } \int_{C_1} \vec{F} \cdot d\vec{r} = -\int_{-C_1} \vec{F} \cdot d\vec{r} = -\int_0^1 (0, t - t^3, t - t^3) \cdot (-2t, 0, 1)\, dt$$

$$= \int_0^1 t^3 - t \, dt = t^4/4 - t^2/2 \Big|_{t=0}^{1} = -\frac{1}{4}$$

Hence $\int_C \vec{F} \cdot d\vec{r} = \sum \int_{C_i} \vec{F} \cdot d\vec{r} = 0 + 0 + 0 - \frac{1}{4} = \boxed{-\frac{1}{4}}$

(c) Compute $\operatorname{curl} \mathbf{F}$. **(2 pts)**

$$\operatorname{curl} \vec{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & xyz & xyz \end{vmatrix} = (xz - xy, -yz, yz)$$

(d) Use Stokes' Theorem to compute the flux of $\operatorname{curl} \mathbf{F}$ through the surface S where the normals point out from the origin. **(3 pts)**

$$\iint (\operatorname{curl} \vec{F}) \cdot \vec{n} \, dA = -\int_C \vec{F} \cdot d\vec{r} = \frac{1}{4}$$

since orient of C doesn't mesh with $\vec{n}$.

(e) Give two distinct reasons why the vector field $\mathbf{F}$ is *not* conservative. **(2 pts)**

$\operatorname{curl} \vec{F} \neq 0$ and C is a closed curve with $\int_C \vec{F} \cdot d\vec{r} \neq 0$.