

Lecture 7: Limit Laws (14.2), Continuity (14.3),
and Partial Derivatives (14.3)

Last time: Limits: definition and examples.

Limit Laws: $f, g: \mathbb{R}^2 \rightarrow \mathbb{R}$.

(a) $\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) + g(\vec{x}) = \lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) + \lim_{\vec{x} \rightarrow \vec{a}} g(\vec{x})$

Provided both limits exist.

(b) $\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x})g(\vec{x}) = \left(\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x})\right)\left(\lim_{\vec{x} \rightarrow \vec{a}} g(\vec{x})\right)$

(c) $\lim_{\vec{x} \rightarrow \vec{a}} \frac{f(\vec{x})}{g(\vec{x})} = \frac{\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x})}{\lim_{\vec{x} \rightarrow \vec{a}} g(\vec{x})}$

Same as plus
 $\lim_{\vec{x} \rightarrow \vec{a}} g(\vec{x}) \neq 0$.

Ex:

$$\lim_{(x,y) \rightarrow (1,2)} \frac{x + x^2}{y} = \frac{\lim_{(x,y) \rightarrow (1,2)} x + x^2}{\lim_{(x,y) \rightarrow (1,2)} y}$$

$$= \frac{\lim_{(x,y) \rightarrow (1,2)} x + \left(\lim_{(x,y) \rightarrow (1,2)} x\right)^2}{2} = \frac{1 + 1^2}{2} = 1$$

[Notice this is also what you'd get by plugging in $(x,y) = (1,2)$]

Continuity: $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous at \vec{a}

if $\lim_{\vec{x} \rightarrow \vec{a}} f(\vec{x}) = f(\vec{a})$

Some Meanings:

(a) Can eval. limit by plugging in

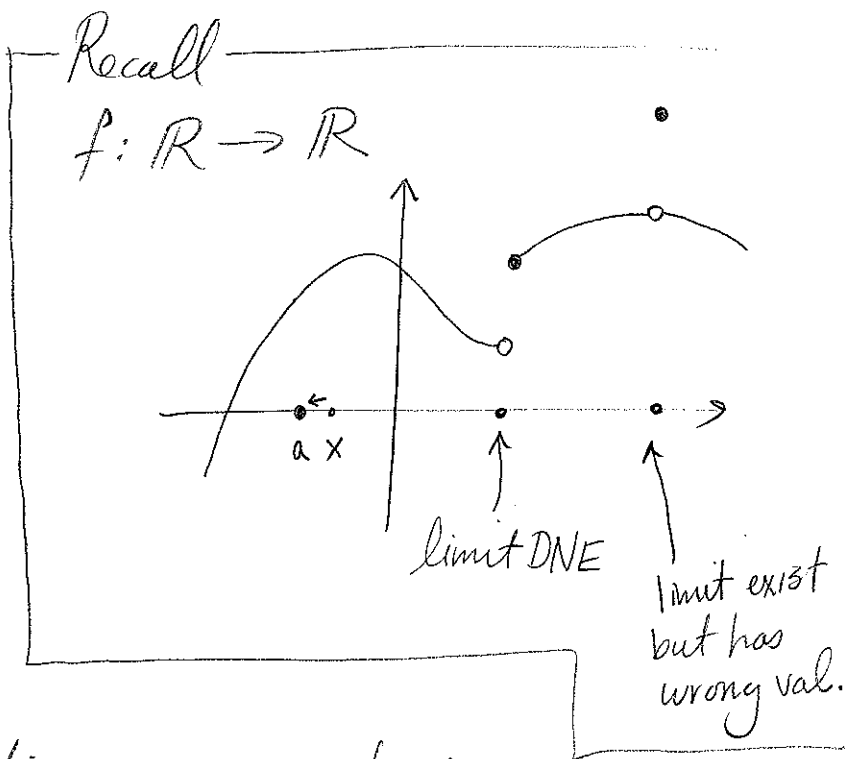
$f(x,y) = \frac{x+x^2}{y}$ is continuous everywhere [it turns out], so

$\lim_{(x,y) \rightarrow (1,2)} f(x,y) = f(1,2) = 1$. [Matches what we just got.]

(b) $f(\vec{a} + \vec{h}) = f(\vec{a}) + E(\vec{h})$ where $\lim_{\vec{h} \rightarrow \vec{0}} E(\vec{h}) = 0$.

Most, but not all, functions you'll encounter "in nature" are continuous. For example, functions built from other cont. functions via

$+$, \times , \div (not by 0), composition (e.g. $\sqrt{x^2 + 2y}$)



Ex: $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by

$$f(x,y) = \begin{cases} \frac{x^2}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

is continuous. At $(x,y) \neq (0,0)$ this is because it is built up from continuous pieces. At $(0,0)$ need to check directly that

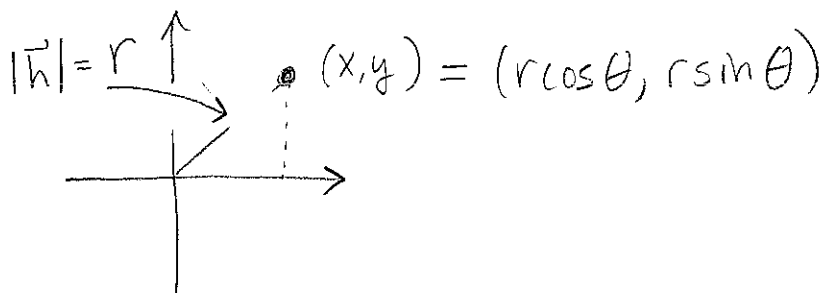
$$\textcircled{\star} \lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{\sqrt{x^2+y^2}} = 0 = f(0,0)$$

WARNING: For fns from $\mathbb{R}^2 \rightarrow \mathbb{R}$ there's no L'Hopital's Rule!

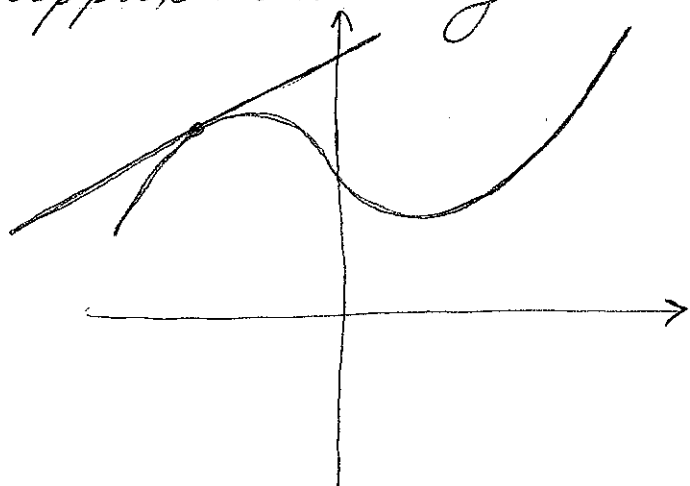
Reason for $\textcircled{\star}$: let $\epsilon > 0$, take $\delta = \epsilon$.

If $\vec{h} = (x,y)$ sat $0 < |\vec{h}| < \delta$, then

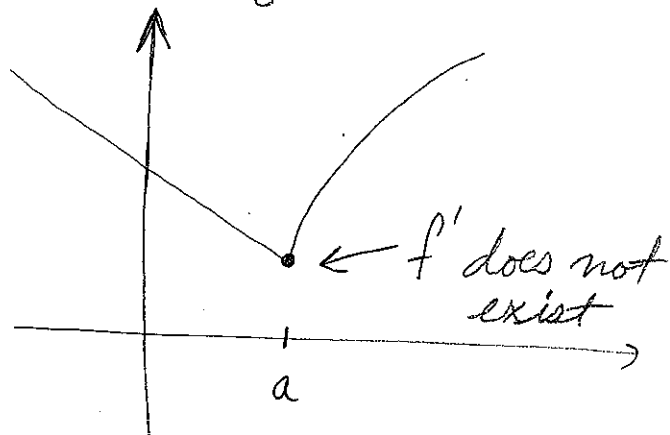
$$\left| \frac{x^2}{\sqrt{x^2+y^2}} \right| = \frac{|r^2 \cos^2 \theta|}{|r|} = r \cos^2 \theta \leq r = |\vec{h}| < \delta.$$



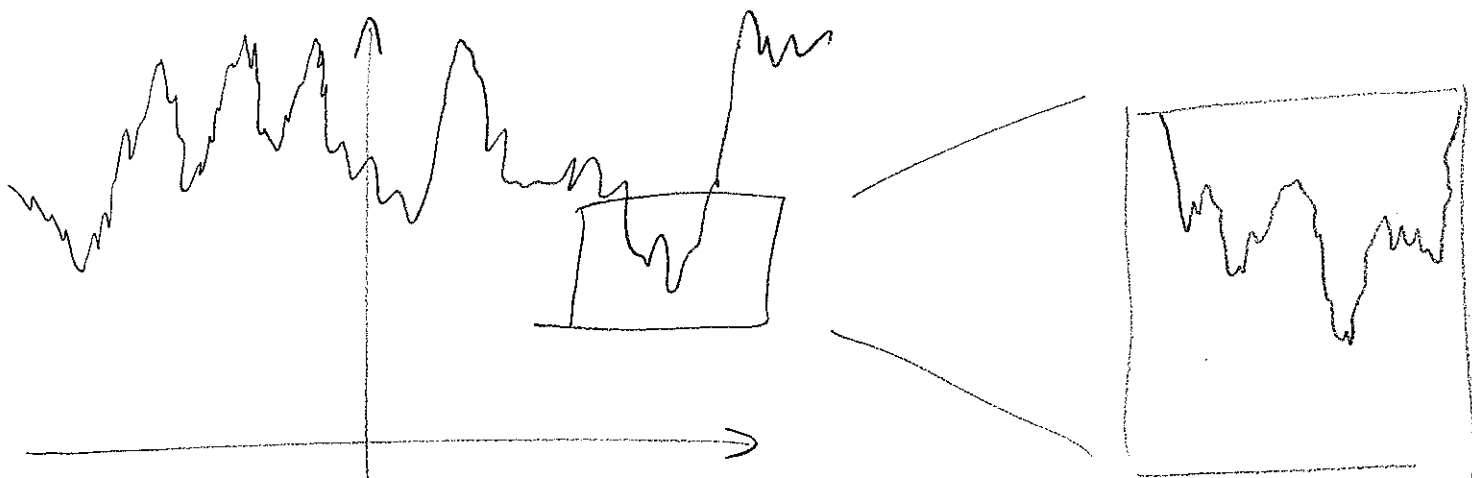
Derivatives: For $f: \mathbb{R} \rightarrow \mathbb{R}$ this is about approximations by lines



Can't always do:

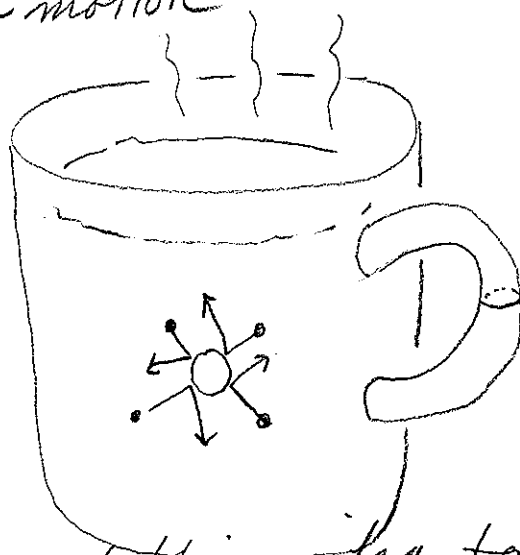


Or even do anywhere:



Examples: Stock market; Brownian motion

Think dust moving in sunlight
Brown (19th cent) observed with pollen moving on the surface of water. Einstein (1905) brought to attention of physicists. 2000 years earlier, the Roman Lucretius used this idea to

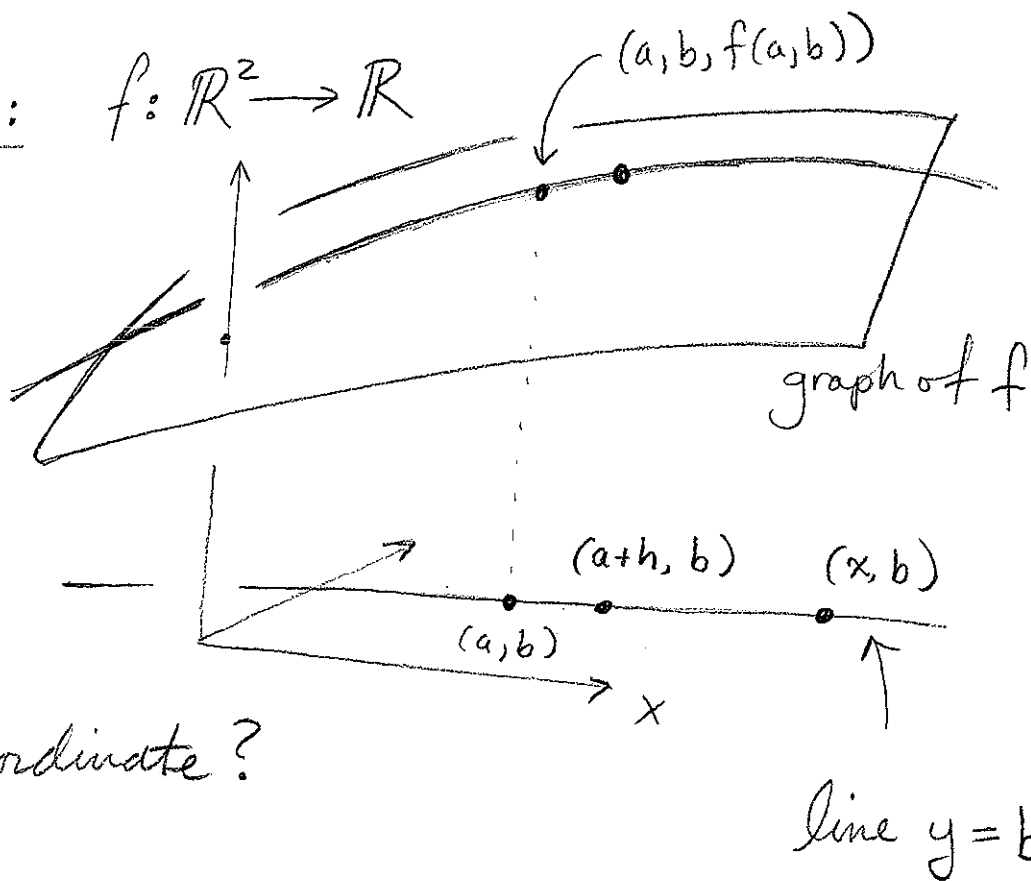


argue for the existence of molecules...

In this class we will work almost exclusively with fns that have derivatives.

Partial derivatives: $f: \mathbb{R}^2 \rightarrow \mathbb{R}$

At what rate
does f change
if we start
at (a, b) and
vary the x -coordinate?



$$\frac{\partial f}{\partial x}(a, b) = \lim_{h \rightarrow 0} \frac{f(a+h, b) - f(a, b)}{h}$$

Compare

$$f'(a) = \frac{df}{dx}(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

Easy to compute: Just view y as a constant and differentiate with respect to x .

Ex: $f(x,y) = x^2 + xy + y^2$

$$\frac{\partial f}{\partial x} = 2x + y + 0 \quad \frac{\partial f}{\partial x}(3,1) = 7$$

Can also look at the rate of change in the y -direction: [View x as a constant]

Ex: $\frac{\partial}{\partial y} ((x+y) \sin(xy))$

$$= \left(\frac{\partial}{\partial y} (x+y) \right) \sin(xy) + (x+y) \frac{\partial}{\partial y} (\sin(xy))$$

$$= \sin xy + (x+y) \cos(xy) x$$

Other notation:

$$\frac{\partial f}{\partial x}(a,b) = \left(\frac{\partial}{\partial x} f\right)(a,b) = f_x(a,b) = D_1 f(a,b)$$

Can take higher order partials

$$\frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2}{\partial y \partial x} f = f_{yx}$$

$$\frac{\partial^2}{\partial y \partial x} (x^2 + xy + y^2) = \frac{\partial}{\partial y} (2x + y) = 1$$

Next time tangent planes.