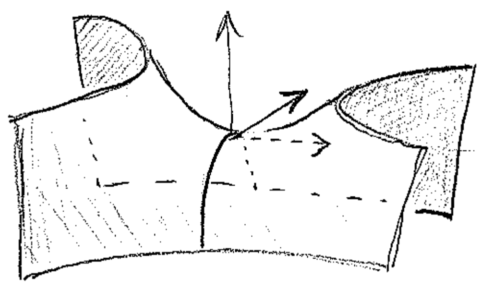
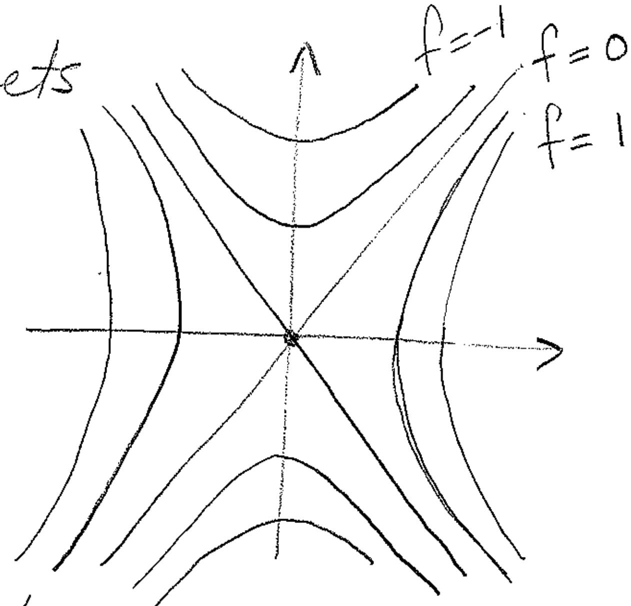
Lecture 6: Level sets in 3^d (14.1), quadric surfaces (12.6), review of limits (14.2)

Last time: $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x^2 - y^2$

Graph

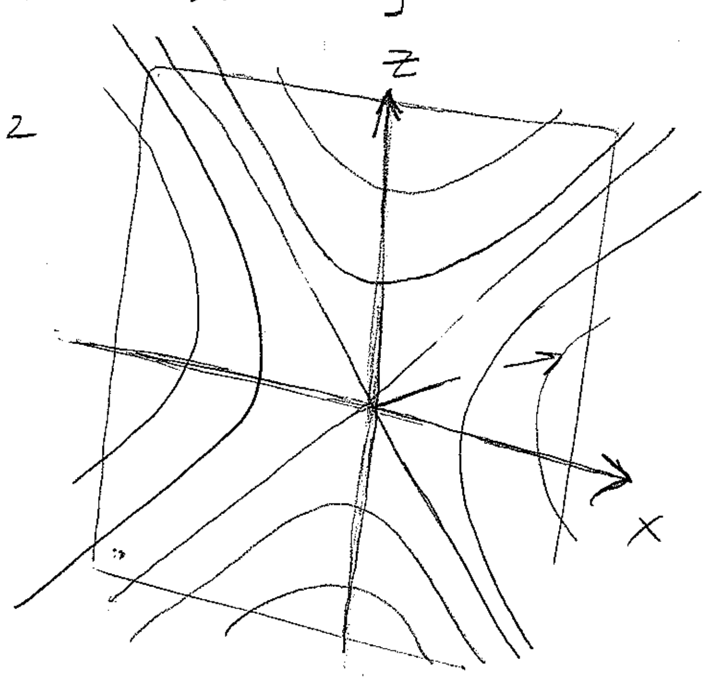


Level sets



For $f: \mathbb{R}^3 \to \mathbb{R}$ can't draw the graph (in $\mathbb{R}^4$) but can still look at level sets.

[Did $f(x,y,z) = x^2 + y^2 + z^2$ last time]



Ex: $f(x,y,z) = x^2 + y^2 - z^2$

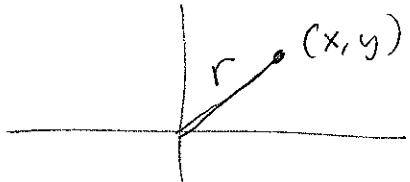
First, look at xz-plane

$$f(x,0,z) = x^2 - z^2$$

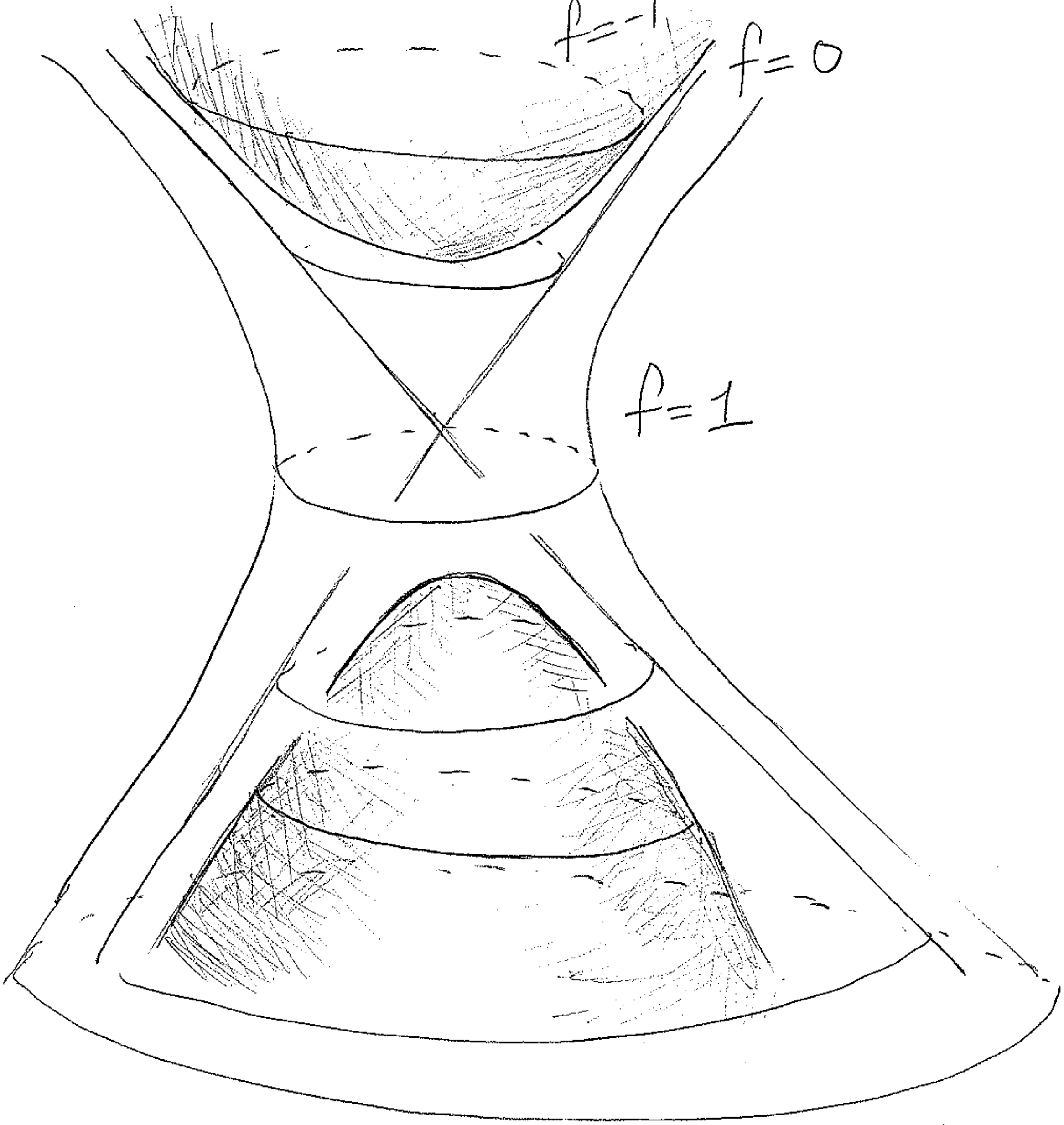
so the level sets

in this plane look like what we had on Wed.

Also, as $x^2+y^2=r^2$



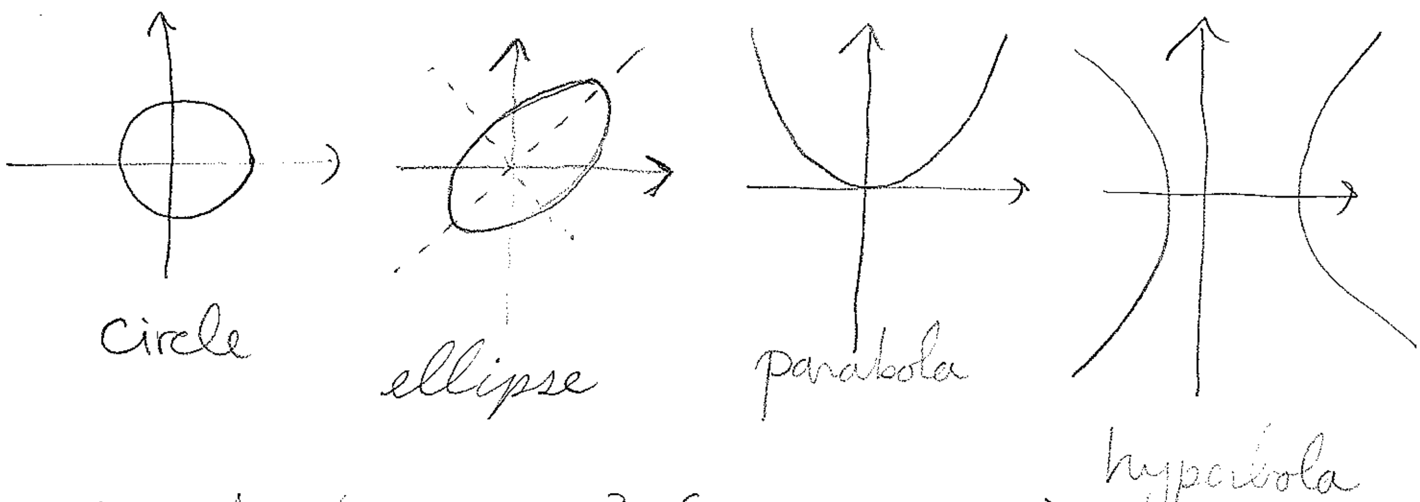
we have $f(x,y,z) = r^2 - z^2$ and so each level set is symmetric about z-axis:



[These level sets are all examples of quadric surfaces.]

Conic Sections: Solutions in $\mathbb{R}^2$

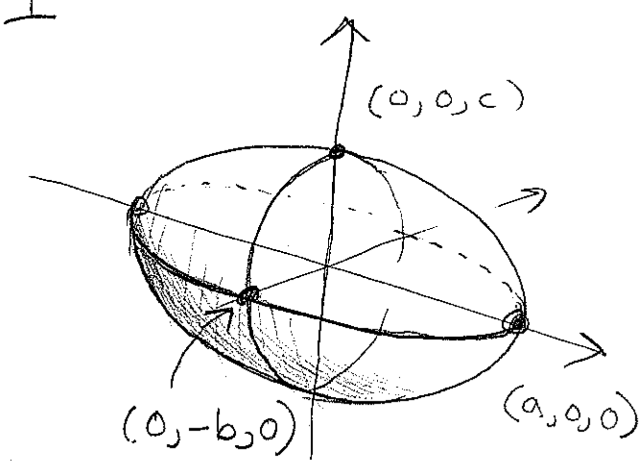
of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$



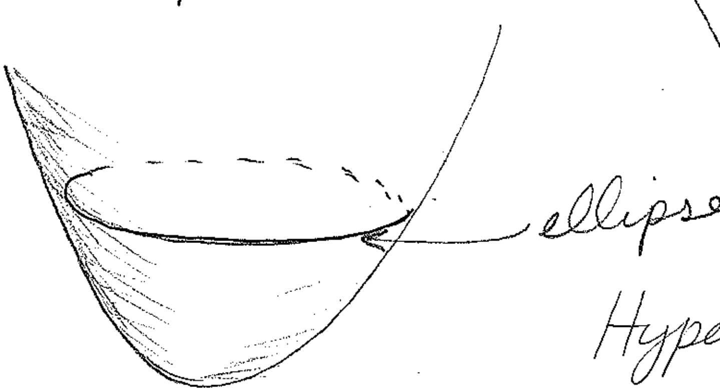
Quadric Surfaces in $\mathbb{R}^3$ (Section 12.6)

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

Ex: Ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$



Elliptic parabolid:



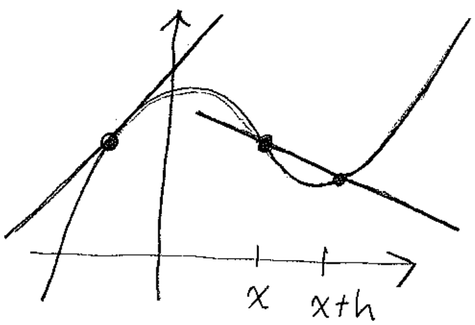
$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Hyperbolic parabolid:

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

The other quadric surfaces are the double cone and the hyperboloids of 1 and 2 sheets that we saw at the start.

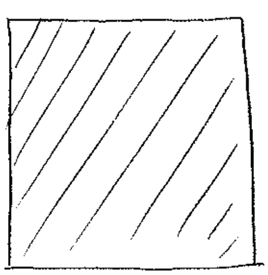
<u>Limits</u> (14.2) [To talk about derivatives, first need to understand limits of fns of several variables.]



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

[Can take different perspectives on limits — I'll focus on them as a way of estimating error.]

Suppose we want to fabricate a square with area $1\,m^2$; and it comes back with sides of length $1+h$



$1+h$

$1+h$

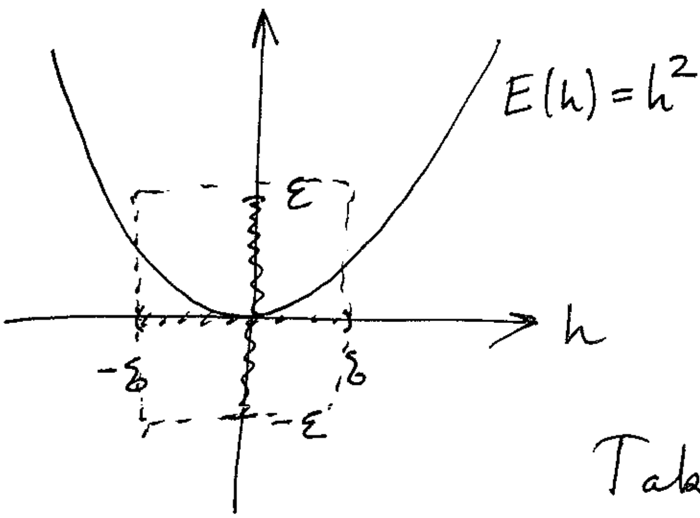
$$\text{Error} = \begin{matrix}\text{actual}\\ \text{area}\end{matrix} - 1 = (1+h)^2 - 1$$
$$= h^2 + 2h$$

<u>Q</u>: If we want error $< \frac{1}{10}$, to what tolerance do we need to make the square?

Consider $E: \mathbb{R} \to \mathbb{R}$ (an "error function")

We say

$\lim_{h \to 0} E(h) = 0$ if given $\varepsilon > 0$ we can always find $\delta > 0$ so that whenever $0 < |h| < \delta$ we have $|E(h)| < \varepsilon$



View as challenge-response process.

Ex: $E(h) = h^2$ $\quad \varepsilon = 1/10$.

Take $\delta = 1/4$. If $|h| < \delta = \frac{1}{4}$ then $|E(h)| = |h^2| = |h|^2 < \frac{1}{16} < \frac{1}{10}$.

Ex: $\varepsilon = 1/100$ $\quad \delta = \boxed{\text{Audience Response}}$

$\varepsilon = 1/1000$ $\quad \delta = \boxed{\text{———— " ————}}$

Claim: $\lim_{h\to 0} h^2 = 0$

Reason: If you give me $\varepsilon > 0$, I'll take $\delta = \sqrt{\varepsilon}$. Then if $|h| < \delta$ we have

$$|h^2| = |h|^2 < \delta^2 = \varepsilon.$$

Ex: $E(h) = 2h + h^2$ Know $\lim_{h\to 0} 2h + h^2 = 0$

Given $\varepsilon = 1/10$ take $\delta = \frac{1}{100}$.

If $|h| < \delta$, then

$$|2h + h^2| \le 2|h| + |h|^2 < 2\cdot\frac{1}{100} + \frac{1}{10{,}000} < \frac{3}{100} < 1/10 = \varepsilon.$$

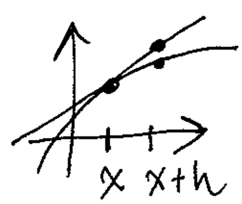
In general, say

$$\lim_{x\to a} f(x) = c$$

if $f(a+h) = c + E(h)$

where $\lim_{h\to 0} E(h) = 0$.

Differentiability:



$f(x+h) = f(x) + f'(x)h + E(h)$

where $E(h)$ is <u>really</u> small, i.e.

$$\lim_{h\to 0} \frac{E(h)}{h} = 0.$$