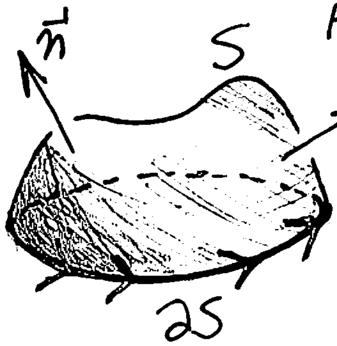
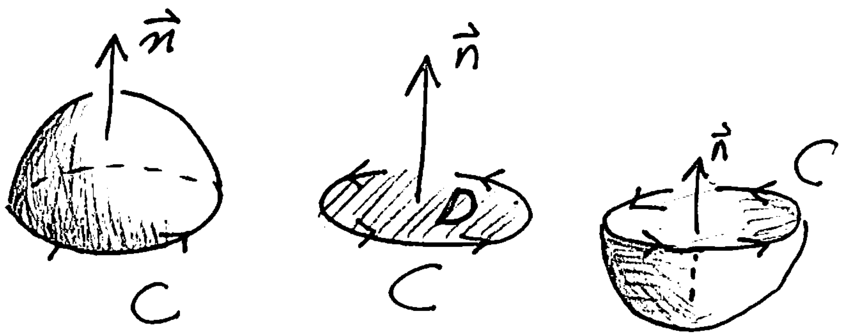
Lecture 40: More on Stokes' Thm

Last time: Stokes' Thm: S surface in $\mathbb{R}^3$, $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ vector field. Then



$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F}) \cdot \vec{n} \, dA$$

Ex: $\vec{F} = (-y, x, yz)$ $\quad$ $\text{curl } \vec{F} = (z, 0, 2)$



$$\iint_S (\text{curl } \vec{F}) \cdot \vec{n} \, dA = 2\pi = \int_C \vec{F} \cdot d\vec{r} \quad \text{for all of these!}$$

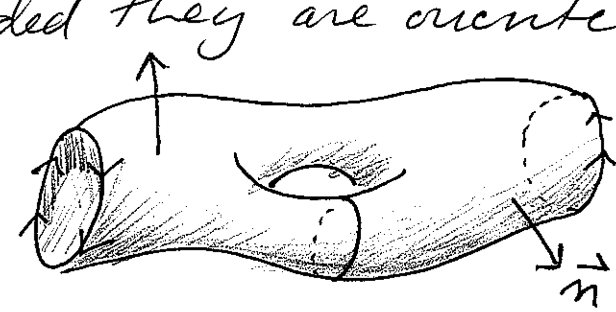
[Takes some getting used to, is really just Green's Thm / 2d Divergence Thm in disguise...]

Check the easy one: $\displaystyle\iint_D (\text{curl } \vec{F}) \cdot \vec{n} \, dA$

$$= \iint_D (z, 0, 2) \cdot (0, 0, 1) \, dA = \iint_D 2 \, dA = 2 \, \text{Area}(\bigcirc)$$

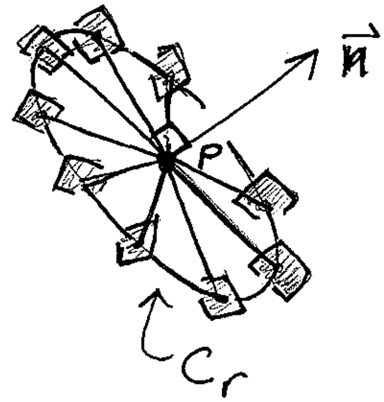
$$= 2\pi$$

Note: Also works when S has several boundary components, [provided they are oriented correctly.]



Understanding Curl: Consider a small paddle wheel at p, of radius r.

$\vec{F}$ = fluid flow

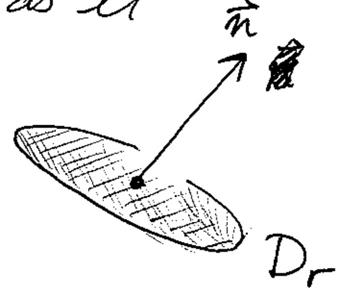


Q: How fast does it spin?

A: $\omega = \frac{1}{2\pi r^2} \int_{C_r} \vec{F} \cdot d\vec{r}$

$\boxed{\text{units} = \frac{\text{radians}}{\text{time}}}$

Plausible: ⓐ Want tangential component of $\vec{F}$ as it hits the paddles.
ⓑ Looks like an average (almost).



Stokes says:

$$\omega = \frac{1}{2\pi r^2} \iint_{D_r} (\text{curl}\,\vec{F}) \cdot \vec{n} \; dA$$

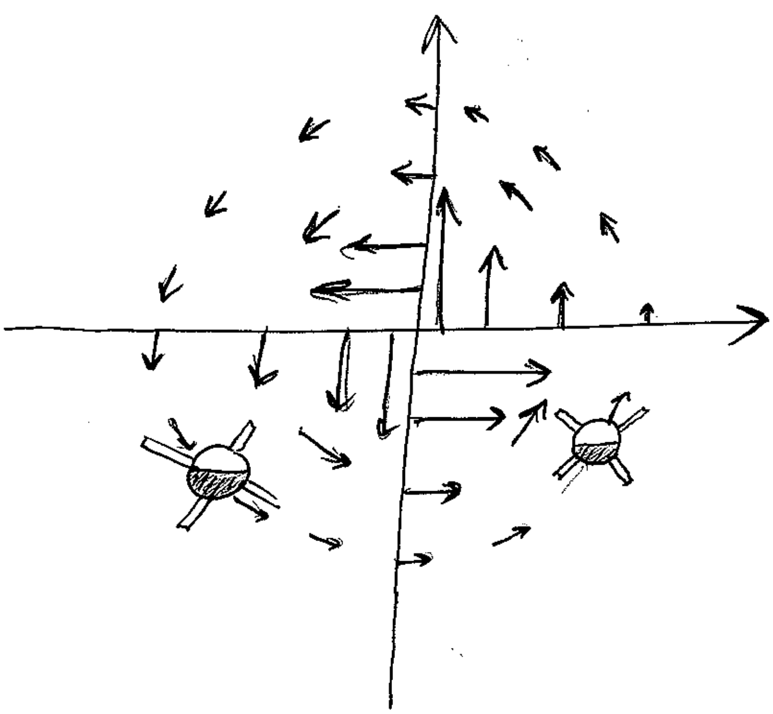
$$= \frac{1}{2}\left(\frac{1}{\text{Area}(D_r)} \iint_{D_r} (\text{curl } \vec{F}) \cdot \vec{n} \, dA\right)$$

Taking $r \to 0$ get: $\omega = \frac{1}{2}(\text{curl } \vec{F}) \cdot \vec{n}$

So the rate of rotation is fastest the direction of $\text{curl } \vec{F}(p)$ and then $\omega = \frac{1}{2}|\text{curl } \vec{F}|$.

<u>Note</u>: A vector field where $\text{curl } \vec{F} = \vec{0}$ everywhere are called <u>irrotational</u>.

<u>Ex</u>: $\vec{F} = \frac{1}{x^2+y^2}(-y, x, 0)$ has $\text{curl } \vec{F} = \vec{0}$ except at $(0,0)$ where it's not defined.



Experimentally, a draining tub is an irrotational flow!

Conservative Vector Fields: $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$

is conservative if $\vec{F} = \nabla f$ for some $f: \mathbb{R}^n \to \mathbb{R}$

Ex: $\vec{F} = (x, y)$ is conservative since

$$= \nabla\left(\tfrac{1}{2}(x^2+y^2)\right)$$

$\vec{F} = (-y, x)$ is not conservative since

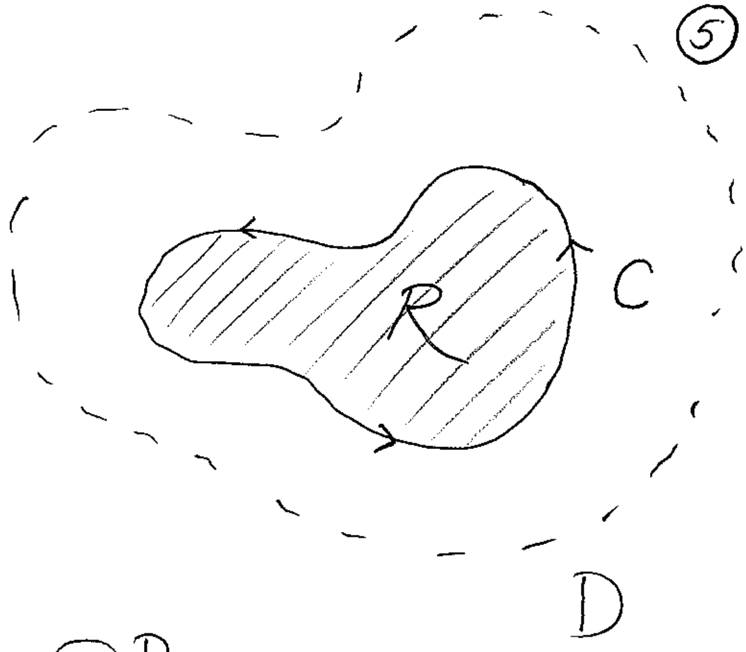
P, Q $$\frac{\partial Q}{\partial x} = 1 \neq -1 = \frac{\partial P}{\partial y}$$

Thm A: $\vec{F}$ on a connected set D in $\mathbb{R}^n$ is conservative if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve C.

Thm B: If D [in $\mathbb{R}^2$] is simply connected (no holes) then $\vec{F} = (P, Q)$ is conservative if and only if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$.

Missing Link: If $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ then $\int_C \vec{F} \cdot d\vec{r} = 0$ for each closed curve C. Hence $\vec{F}$ is conservative by Theorem A.

Reason: As D has no holes, the curve C is the boundary of a region R [where $\vec{F}$ makes sense.] Then



$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{=0!} \, dA = 0.$$

Next time: What is Thm B for $\mathbb{R}^3$?

Suppose $\vec{F} = \nabla f = (f_x, f_y, f_z)$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \partial f/\partial z \end{vmatrix} = \Big(\underbrace{\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}}_{0}, \ 0, \ 0 \Big)$$

Q: Is this enough?