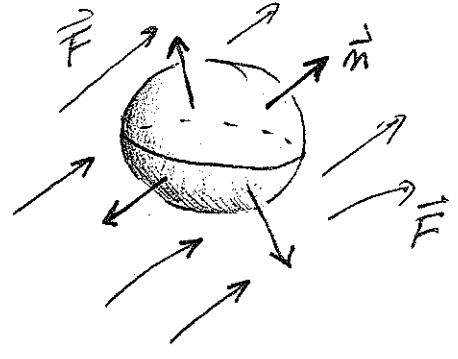


Lecture 39: Stokes Thm (16.8)

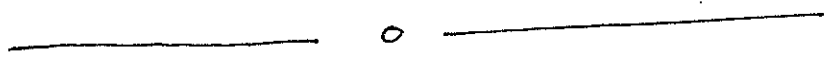
Last time: S surface in \mathbb{R}^3 , $\vec{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ a vector field

Flux = $\iint_S (\vec{F} \cdot \vec{n}) dA$ where \vec{n} is a unit normal vector field.

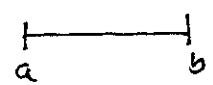



Divergence Thm: D a region in \mathbb{R}^3 , $\vec{F}: D \rightarrow \mathbb{R}^3$ a vector field. Then


$$\iint_{\partial D} (\vec{F} \cdot \vec{n}) dA = \iiint_D \text{div } \vec{F} dV$$

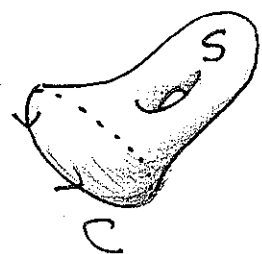


Integral Thms:

1-d: 
 $f(b) - f(a) = \int_a^b f'(t) dt$

1-d in 3-d: 
 $f(B) - f(A) = \int_C \nabla f \cdot d\vec{r}$

2-d: Green's Thm
 $\vec{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ 
 $\int_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} \right) dA$
where $\vec{F} = (P, Q)$.

2-d in 3d: Stokes Thm
 $\vec{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ 
 $\int_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F}) \cdot \vec{n} dA$

3-d: Divergence Thm.

[Say something about how these are really the same...]

Curl: $\vec{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ a vector field with $F = (F_1, F_2, F_3)$

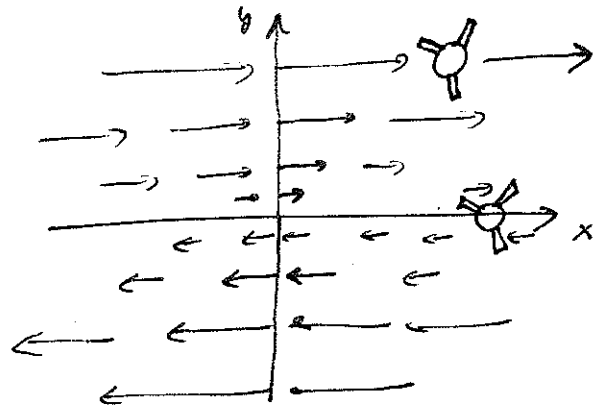
$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Another vector field.

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \vec{k}$$

Ex: $\vec{F} = (y, 0, 0)$

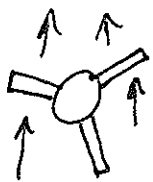
$\text{curl } \vec{F} = (0, 0, -1)$



Q: What does curl measure?

depend only on (x, y) .

First, consider $\vec{F} = (\vec{F}_1, \vec{F}_2, 0)$. [which is effectively a vector field on \mathbb{R}^2] Place a small paddle wheel into the flow. As it moves along with the flow,



$|\text{curl } \vec{F}|$ is the rate of rotation

[precisely 2. (angular velocity)]. Also

$$\text{curl } \vec{F} = \left(0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

where is > 0 if rotation is anticlockwise.

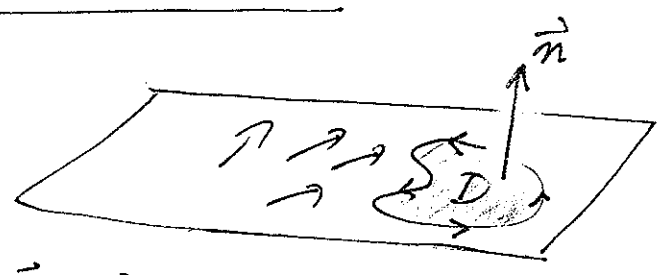
[Will discuss the general meaning of $\text{curl } \vec{F}$ next time.]

Stokes Thm: S surface in \mathbb{R}^3 with boundary curve C .

$\vec{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ a vector field.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl } \vec{F}) \cdot \vec{n} \, dA$$

Relation to Green's Thm:

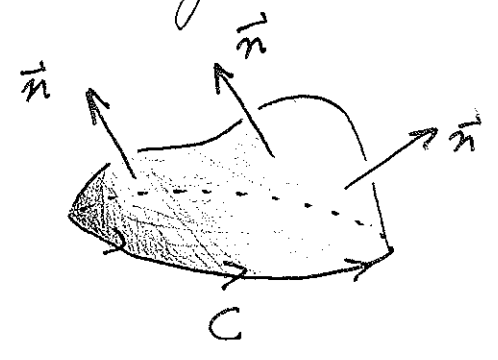


$$\vec{F} = (\vec{F}_1, \vec{F}_2, 0)$$

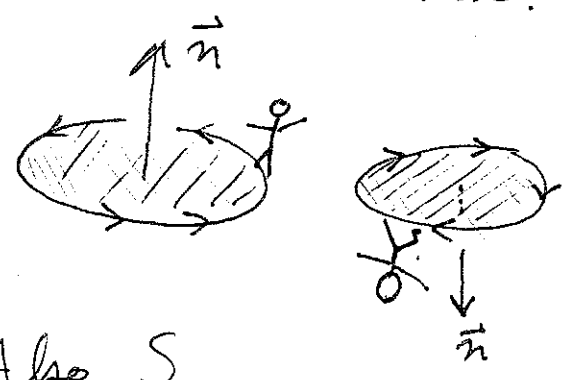
$$\vec{n} = \vec{k}$$

So

$$\begin{aligned} (\text{curl } \vec{F}) \cdot \vec{n} &= \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \\ &= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{aligned}$$



C oriented so that S is to your left as you walk around C with head pointing in the normal direction.

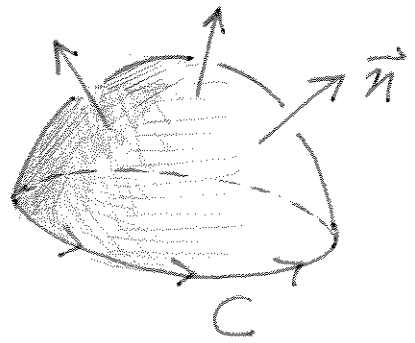


Also, S is orientable.

Example: $S =$ upper unit hemisphere

$\vec{n} =$ outward normal

$$\vec{F} = (-y, x, yz)$$



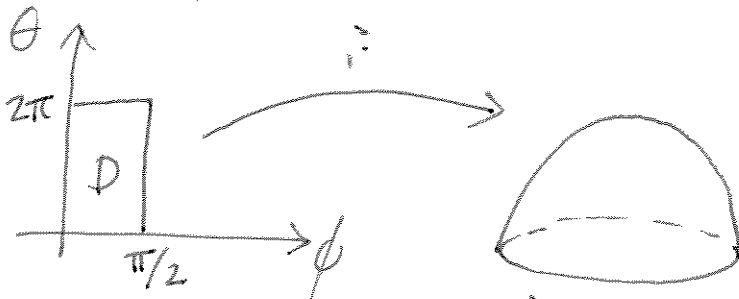
$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & yz \end{vmatrix} = (z, 0, 2)$$

$$\iint_S (\text{curl } \vec{F}) \cdot \vec{n} \, dA = \iint_S (z, 0, 2) \cdot (x, y, z) \, dA$$

$$= \iint_S \underbrace{xz + 2z} \, dA = \iint_S 2z \, dA$$

integrates to 0
by symmetry;

Parameterize S :



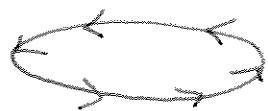
$$\vec{r}(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$$

$$dA = \sin \phi \, d\theta \, d\phi$$

$$\iint_S 2z \, dA = \int_0^{2\pi} \int_0^{\pi/2} 2 \cos \phi \sin \phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \sin^2 \phi \Big|_{\phi=0}^{\phi=\pi/2} d\theta = \int_0^{2\pi} 1 \, d\theta = \boxed{2\pi}$$

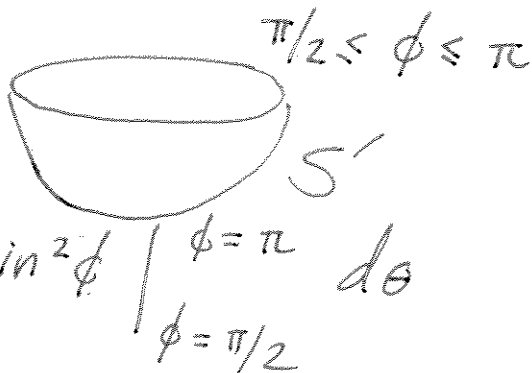
Compare: $\vec{f}(t) = (\cos t, \sin t, 0)$



$$\begin{aligned}\int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \vec{F}(\vec{f}(t)) \cdot \vec{f}'(t) dt \\ &= \int_0^{2\pi} (-\sin t, \cos t, 0) \cdot (-\sin t, \cos t, 0) dt \\ &= \int_0^{2\pi} \sin^2 t + \cos^2 t dt = \int_0^{2\pi} 1 dt = \boxed{2\pi} \checkmark\end{aligned}$$

So Stokes' Thm works in this case!

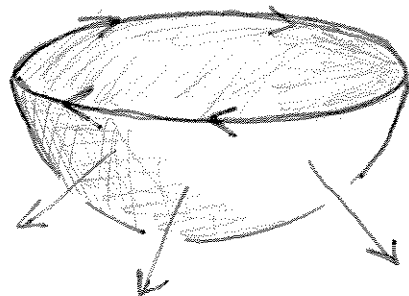
What about the lower hemisphere?



$$\begin{aligned}\iint_{S'} (\text{curl } \vec{F}) \cdot \vec{n} dA &= \dots = \int_0^{2\pi} \sin^2 \phi \Big|_{\phi=\pi/2}^{\phi=\pi} d\theta \\ &= \int_0^{2\pi} -1 d\theta = -2\pi.\end{aligned}$$

But: C is also the boundary of S' , so shouldn't we get $\int_C \vec{F} \cdot d\vec{r} = 2\pi$?

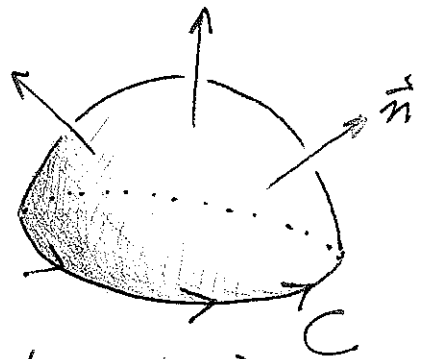
Solution: With outward normal, C gets oriented the other way by S'



Example: $S =$ upper unit hemisphere

$\vec{n} =$ outward normal

$$\vec{F} = (y, xz, 1)$$

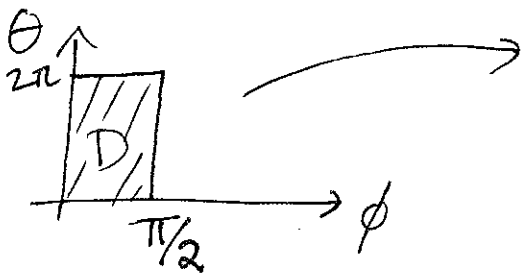


$$\vec{r}(t) = (\cos t, \sin t, 0)$$

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} (\sin t, 0, 1) \cdot (-\sin t, \cos t, 0) dt \\ &= \int_0^{2\pi} -\sin^2 t dt = \boxed{-\pi} \end{aligned}$$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & xz & 1 \end{vmatrix} = (-x, 0, z-1)$$

Parameterize S:



$$dA = \sin \phi d\theta d\phi$$

$$\vec{r}(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$$

$$\iint_S (\text{curl } \vec{F}) \cdot \vec{n} dA = \iint_S (x, 0, z-1) \cdot (x, y, z) dA$$

$$= \iint_S -x^2 + z^2 - z dA = \int_0^{\pi/2} \int_0^{2\pi} (-\sin^2 \phi \cos^2 \theta + \cos^2 \phi - \cos \phi) \sin \phi d\theta d\phi$$

$$= \int_0^{\pi/2} (-\pi \sin^2 \phi + 2\pi \cos^2 \phi - 2\pi \cos \phi) \sin \phi \, d\phi$$

$$-\sin^2 \phi = -1 + \cos^2 \phi$$

$$= \pi \int_0^{\pi/2} (-1 + 3\cos^2 \phi - 2\cos \phi) \sin \phi \, d\phi \quad \begin{array}{l} u = -\cos \phi \\ du = \sin \phi \, d\phi \end{array}$$

$$= \pi \int_{-1}^0 (-1 + 3u^2 + 2u) \, du = \pi \left(-u + u^3 + u^2 \right) \Big|_{-1}^0 = \boxed{-\pi} \checkmark$$

What about the lower hemisphere:

$$\iint_{S'} (\text{curl } \vec{F}) \cdot \vec{n} \, dA$$



$$= \int_{\pi/2}^{\pi} \int_0^{2\pi} (\text{same as before}) \, d\theta \, d\phi = \pi \left(-u + u^3 + u^2 \right) \Big|_{u=0}^{-1} = \boxed{\pi}$$

But C is also the boundary of S' , so shouldn't

we get $\int_C \vec{F} \cdot d\vec{r} = -\pi$?

Solution: With outward normal,

C gets oriented the other way by S' .

