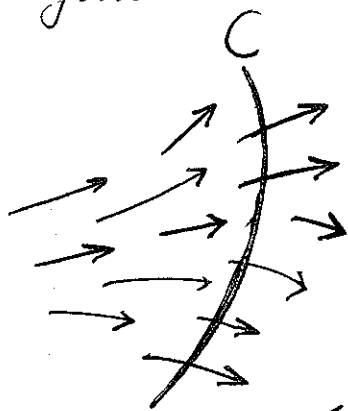


Last time: Flux

$$\vec{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

Reminder:  
Exam Wed



Flux = rate water is crossing  $\vec{F}$

$$= \int_C (\vec{F} \cdot \vec{n}) ds$$

where  $\vec{n}$  is a unit normal vector field along  $C$ .

Today, will use the concept of flux to give a different formulation of Green's Theorem, one that will explain why it works, and will also generalize to  $\mathbb{R}^3$

Divergence:  $\vec{F}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  a vector field with  $\vec{F} = (F_1, F_2)$

$$\text{div } \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \quad \text{which is [Q.] a function } \mathbb{R}^2 \rightarrow \mathbb{R}$$

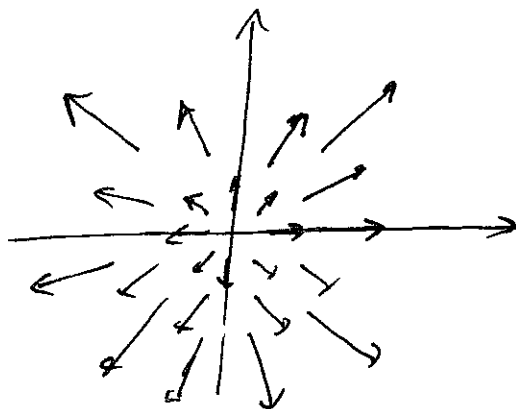
$$\nabla \cdot \vec{F}$$

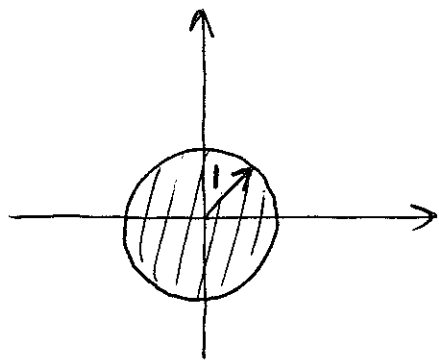
Meaning: [Thinking of  $\vec{F}$  as rep. fluid flow:]

$\text{div } \vec{F}$  = rate of expansion of area under the flow.

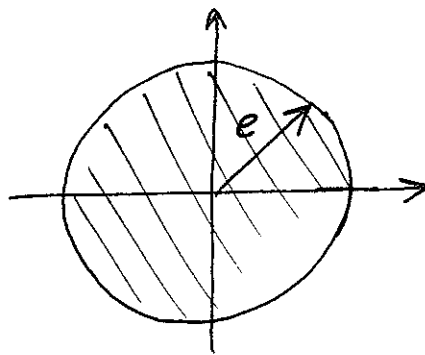
Ex:  $\vec{F} = (x, y)$

Suppose we dye the water inside the unit circle green.



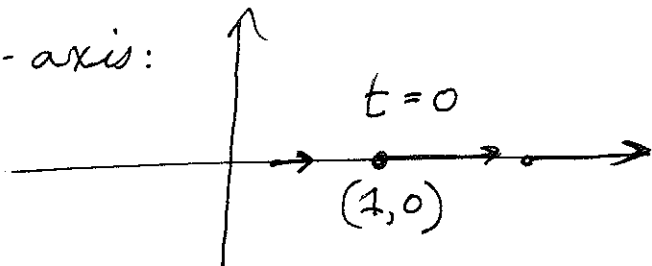


1 unit  
of time



Reason: Consider the flow on the  $x$ -axis:

$$x'(t) = x(t) \Rightarrow x(t) = e^t$$



In general, the follow of  $(x_0, y_0)$  is

given by  $\vec{f}(t) = (x_0 e^t, y_0 e^t)$  since  $\vec{f}'(t) = \vec{f}(t) = \vec{F}(\vec{f}(t))$

So

$$\frac{\text{Green Area}(t)}{\text{Green Area @ } t=0} = \frac{2\pi(e^t)^2}{2\pi} = e^{2t} \quad \begin{array}{l} \sqrt{\text{rate of increase}} \\ \text{in the} \\ \text{green area.} \end{array}$$

Note

$$\text{div } \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 2.$$

match!

In real life, how could this happen?

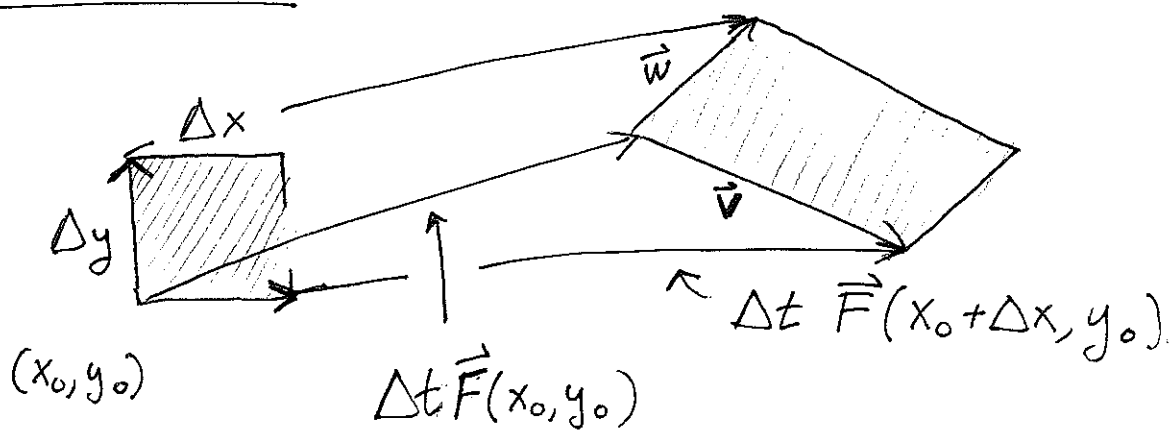
(a) Fluid becomes more/less dense (compressible)

[When modeling airflow over a wing, usually assume  $\text{div } \vec{F} = 0$

(b) Fluid is being added somehow.

if speeds are  
< Mach 0.3 ]

Approx Picture:



$$\vec{v} \approx (\Delta x, 0) + \Delta t \Delta x \vec{F}_x(x_0, y_0) = \left( \Delta x + \Delta t \Delta x \frac{\partial F_1}{\partial x}, \Delta t \Delta x \frac{\partial F_2}{\partial x} \right)$$

$$= \Delta x \left( 1 + \Delta t \frac{\partial F_1}{\partial x}, \Delta t \frac{\partial F_2}{\partial x} \right)$$

$$\vec{w} \approx \Delta y \left( \Delta t \frac{\partial F_1}{\partial y}, 1 + \Delta t \frac{\partial F_2}{\partial y} \right)$$

consider skipping

So the new area is

$$= \Delta x \Delta y \left( 1 + \Delta t (\operatorname{div} \vec{F}) + \Delta t^2 (\text{stuff}) \right)$$

Thus:

$$\frac{\text{new area}}{\text{old area}} \approx 1 + \Delta t (\operatorname{div} \vec{F})$$

and so the rate of expansion is  $(\operatorname{div} \vec{F})$ .

Check units:

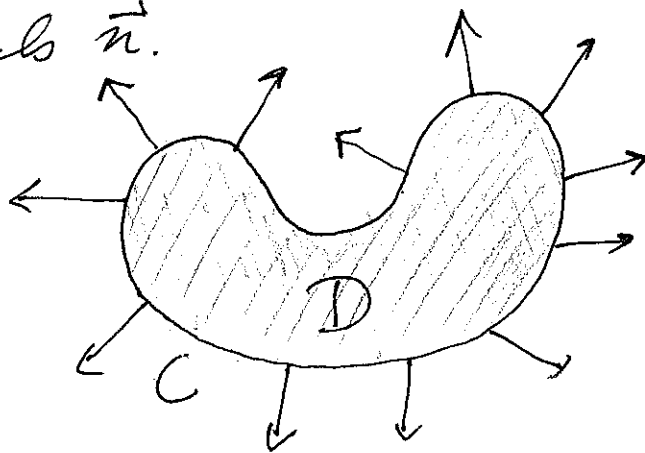
$\vec{F}(x,y)$  has units  $m/s$

$\text{div } \vec{F}$  has units  $1/s$  and so  $1 + \Delta t (\text{div } \vec{F})$  is dimensionless.

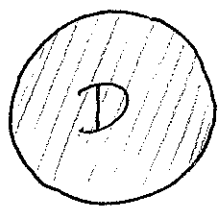
Divergence Thm:  $D$  a region in  $\mathbb{R}^2$  bounded by  $C$ , with outward unit normals  $\vec{n}$ .

Then

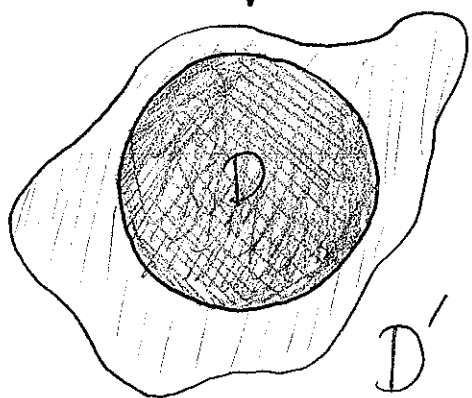
$$\underbrace{\int_C \vec{F} \cdot \vec{n} \, ds}_{\text{Flux}} = \iint_D \text{div } \vec{F} \, dA$$



Reason: After time  $\Delta t$ , the fluid in  $D$  now fills the region  $D'$ . Now



$\Delta t$



$$\frac{\text{Area}(D')}{\text{Area}(D)} \approx 1 + \Delta t \cdot r$$

where  $r = \text{Ave. rate of expansion} = \frac{1}{\text{Area}(D)} \iint_D \text{div } \vec{F} \, dA$

$\Rightarrow$

$$\text{Area}(D') - \text{Area}(D) \approx \Delta t \iint_D \text{div } \vec{F} \, dA$$

Now as the region  $D$  is fixed, the change in area can only be accomplished by fluid crossing  $C$ . The amount that crosses  $C$

in time  $\Delta t$  is  $\approx \Delta t \int_C (\vec{F} \cdot \vec{n}) ds$ . Thus we

$$\text{must have } \int_C \vec{F} \cdot \vec{n} ds = \iint_D \text{div } \vec{F} dA$$


---

How this relates to Green's Theorem:

$$\text{If } \vec{F} = (F_1, F_2) \text{ set } \vec{G} = (-F_2, F_1) = (P, Q)$$

Then

$$\int_C (\vec{F} \cdot \vec{n}) ds = \int_C \vec{G} \cdot d\vec{r} \quad (\star)$$

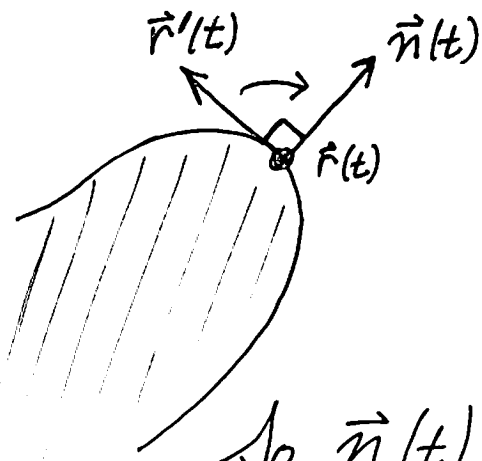
and

$$\text{div } \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

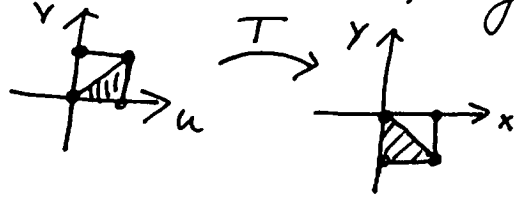
so

$$\iint_D \text{div } \vec{F} dA = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Reason for \*: Take a unit-speed param  $\vec{r}: [a, b] \rightarrow C$ .



Get  $\vec{n}(t)$  from  $\vec{r}'(t)$  by rotating right



$$T(u, v) = (v, -u)$$

So  $\vec{n}(t) = (r_2'(t), -r_1'(t))$  and

$$\int_C \vec{F} \cdot \vec{n} \, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{n}(t) \, dt$$

$$= \int_a^b F_1(\vec{r}(t)) r_2'(t) - F_2(\vec{r}(t)) r_1'(t) \, dt$$

$$= \int_a^b G(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

$$= \int_C G \cdot d\vec{r}$$

Next time: Flux across a surface

