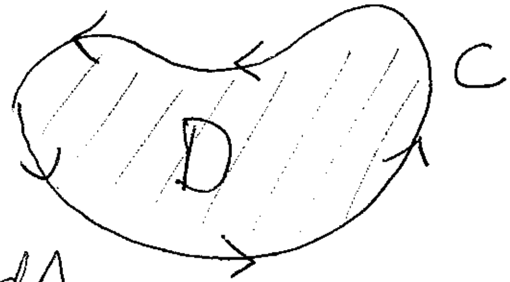
Lecture 33: Surfaces in $\mathbb{R}^3$.



Last time: Green's Thm.

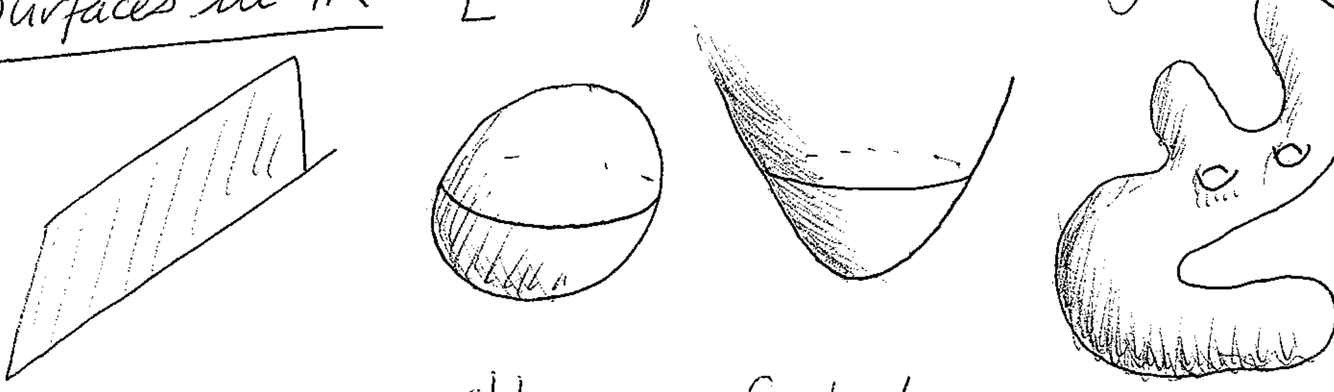
$$\int_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

for any vector field $\vec{F}: D \to \mathbb{R}^2$ with $\vec{F} = (P, Q)$

Query: What happens if we oriented C the other way?

[Last time, I explained why this works ~~in general~~ for $\vec{F} = \frac{1}{2}(-y, x)$. Will discuss the general case later, but for now I'd like surfaces and integration over such $\Longrightarrow$ more analogs of the Fund. Thm. of Calc.]

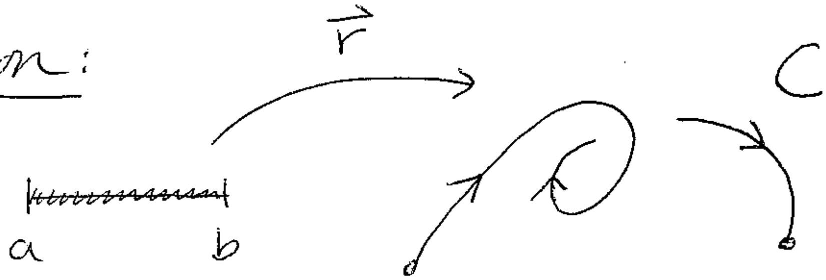
Surfaces in $\mathbb{R}^3$: [Things that look locally like $\mathbb{R}^2$.]



Plane Sphere Graph of $f: \mathbb{R}^2 \to \mathbb{R}$?

Parameterization:

Curves:

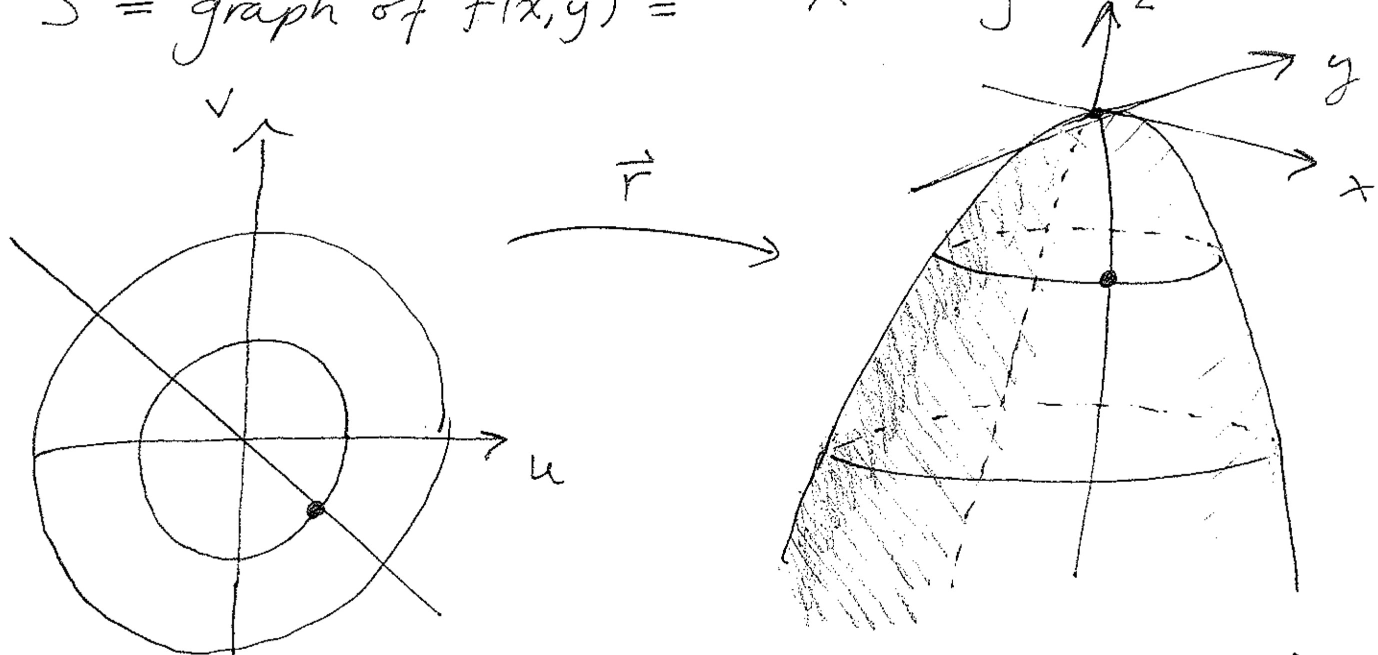


$$\vec{r} : [a,b] \longrightarrow \mathbb{R}^3$$

Surface:

$$\vec{r} : D \longrightarrow \mathbb{R}^3$$

Ex: $S =$ graph of $f(x,y) = -x^2 - y^2$

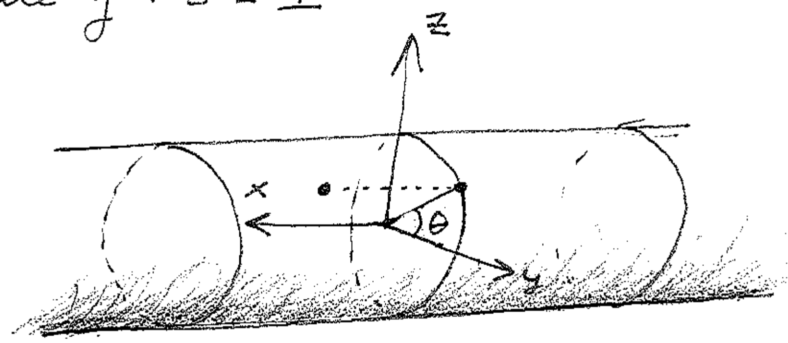


$\vec{r} : \mathbb{R}^2 \to \mathbb{R}^3$ given by $\vec{r}(u,v) = (u, v, -u^2 - v^2)$

Note: $\vec{r}(u,v) = (u, v, f(u,v))$ works to parameterize the graph of any $f : \mathbb{R}^2 \to \mathbb{R}$.

Ex: S = cylinder where $y^2+z^2=1$

To specify a point, need to give:

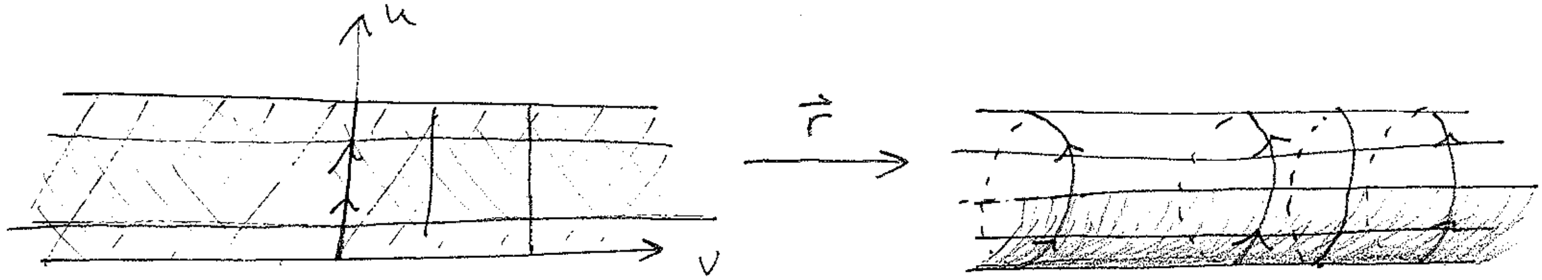


X coor $\longrightarrow u$

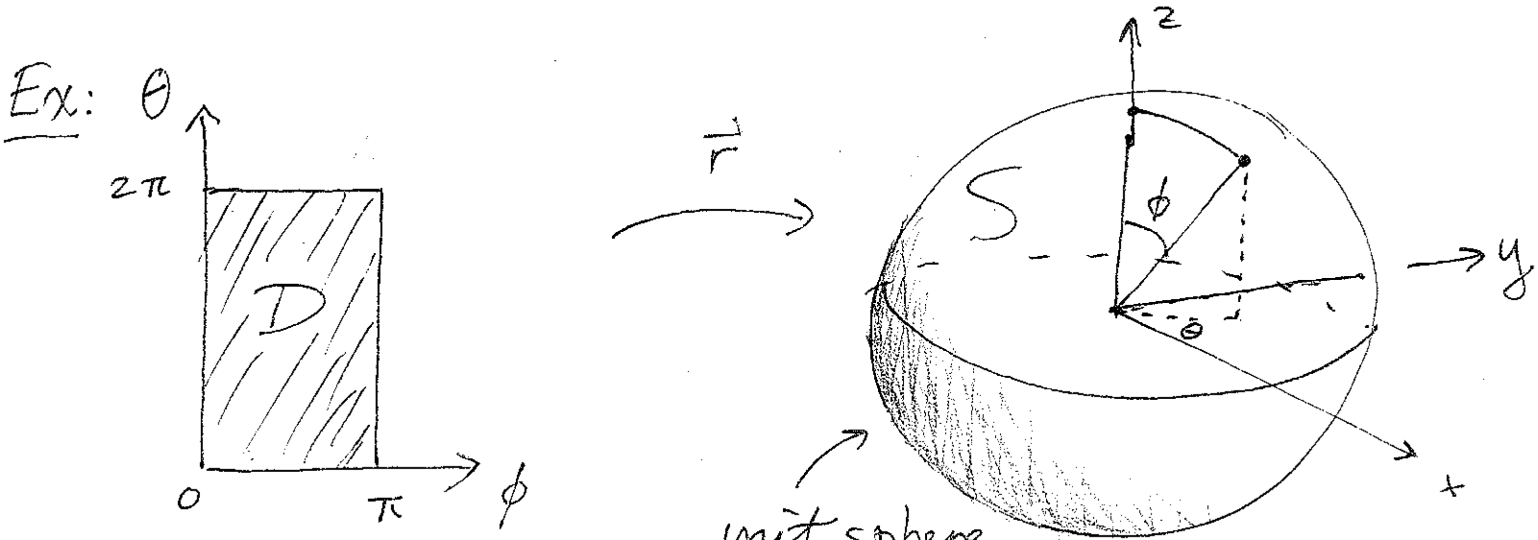
θ angle $\longrightarrow v$

$\vec{r}: D \longrightarrow \mathbb{R}^3$

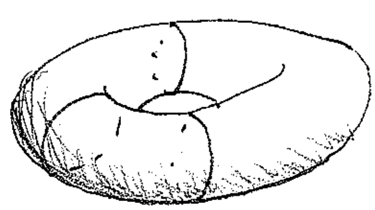
$D = \{0 \leq v \leq 2\pi\}$ $\quad \vec{r}(u,v) = (u, \cos v, \sin v)$



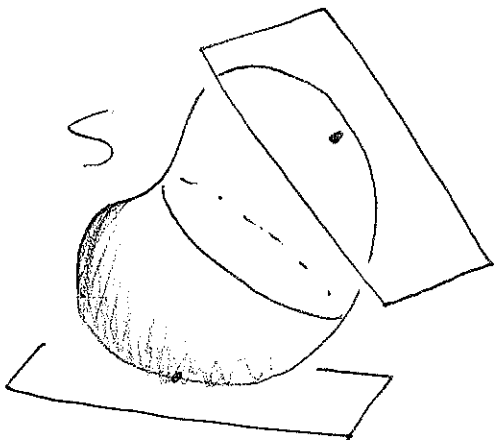
Ex:



$\vec{r}(\phi,\theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$

Many more examples:  ...

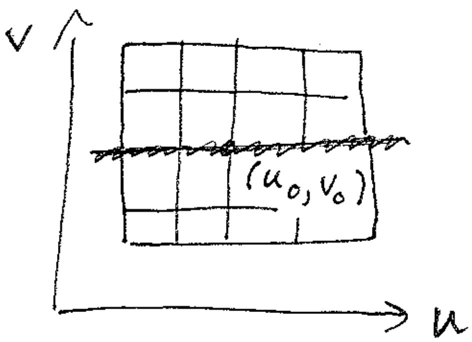
Tangent planes: [Encountered before, for graphs and level sets...]

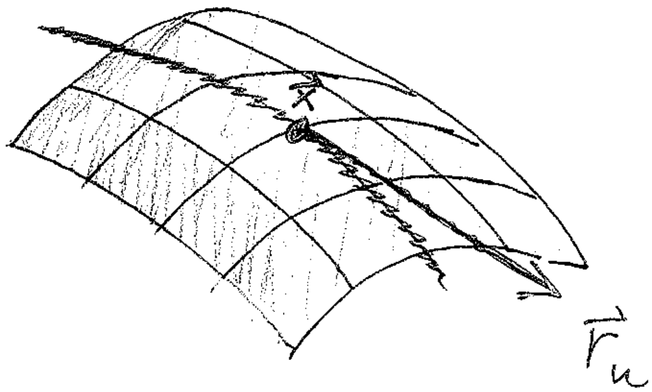


Locally approximate S.

[Compare  .]

Consider a param $\vec{r}: D \to \mathbb{R}^3$





Consider the curve

$$\vec{f}(t) = \vec{r}(u_0 + t, v_0)$$

which passes through $\vec{x}$ at time $t = 0$. Its tangent vector at $t=0$ is

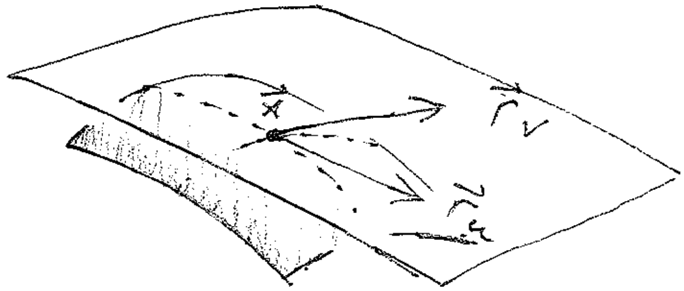
$$\vec{f}'(0) = \frac{d}{dt}\Big(r_1(u_0+t, v_0), r_2(u_0+t, v_0), r_3(u_0+t, v_0)\Big)\Big|_{t=0}$$

$$= \left(\frac{\partial r_1}{\partial u}(u_0, v_0), \frac{\partial r_2}{\partial u}(u_0, v_0), \frac{\partial r_3}{\partial u}(u_0, v_0)\right)$$

$$= \vec{r}_u(u_0, v_0)$$

where $\vec{r}(u,v) = (r_1(u,v), r_2(u,v), r_3(u,v))$

Similarly we have $\vec{r}_v(u_0, v_0)$



Together, $\vec{r}_u$ and $\vec{r}_v$ span the tangent plane to S at $\vec{x}$.

Ex: Find the tangent plane to the unit sphere at $(1/\sqrt{2}, 0, 1/\sqrt{2})$. In terms of

$$\vec{r}(\phi, \theta) = (\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi)$$

this is $\vec{r}(\pi/4, 0)$.

$$\vec{r}_\phi = (\cos\phi\cos\theta,\ \cos\phi\sin\theta,\ -\sin\phi)$$

$$\vec{r}_\phi(\pi/4, 0) = (1/\sqrt{2},\ 0,\ -1/\sqrt{2})$$

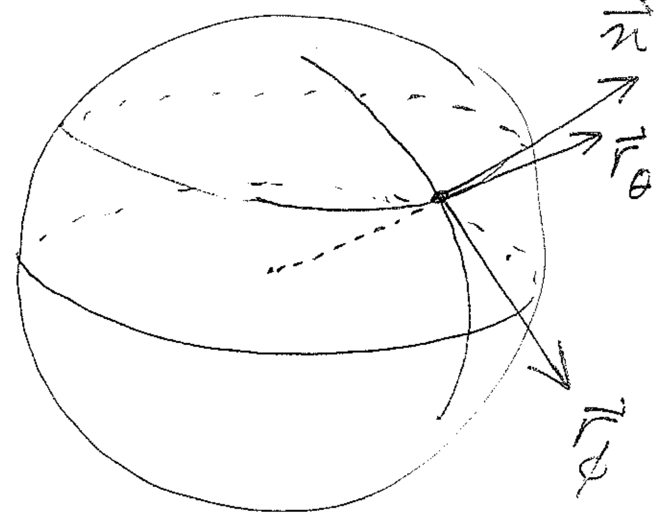
$$\vec{r}_\theta = (-\sin\phi\sin\theta,\ \sin\phi\cos\theta,\ 0)$$

$$\vec{r}_\theta(\pi/4, 0) = (0,\ 1/\sqrt{2},\ 0)$$

So a normal vector is

$$\vec{n} = \vec{r}_\phi \times \vec{r}_\theta$$

$$= \left(1/2, 0, 1/2 \right)$$



which points straight out from the sphere,

as we expect.