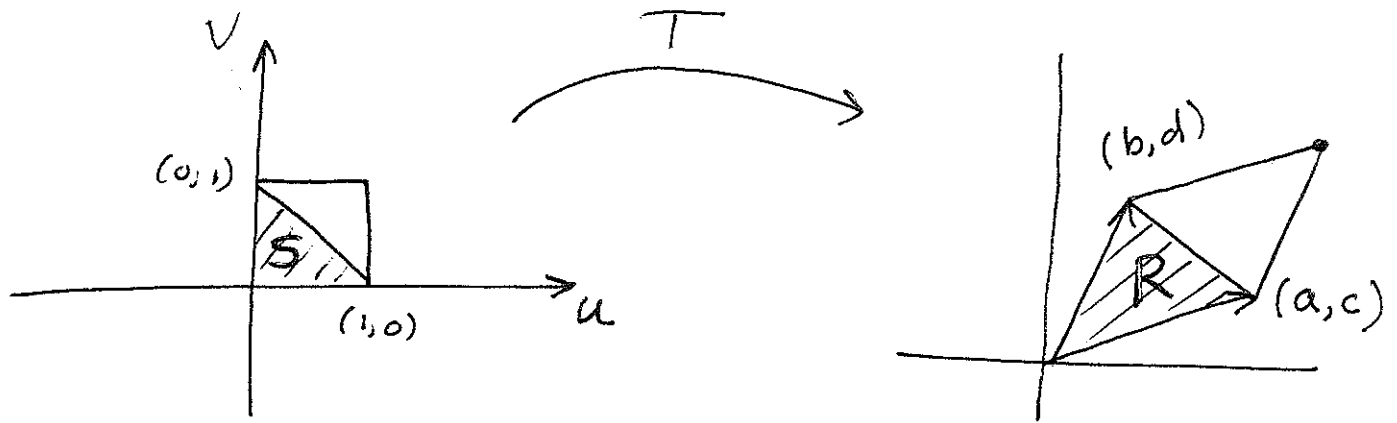


Lecture 31: Changing coordinates II (15.9)

Next time: Section 16.4 (Green's Thm)

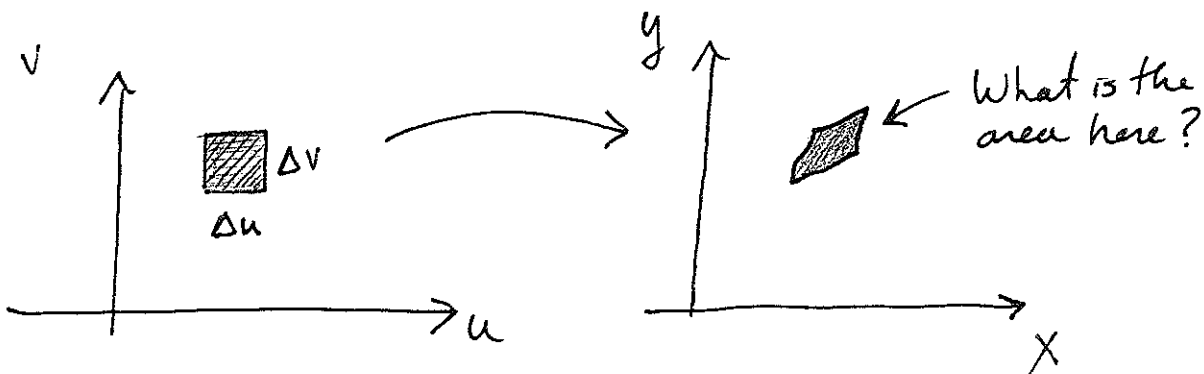
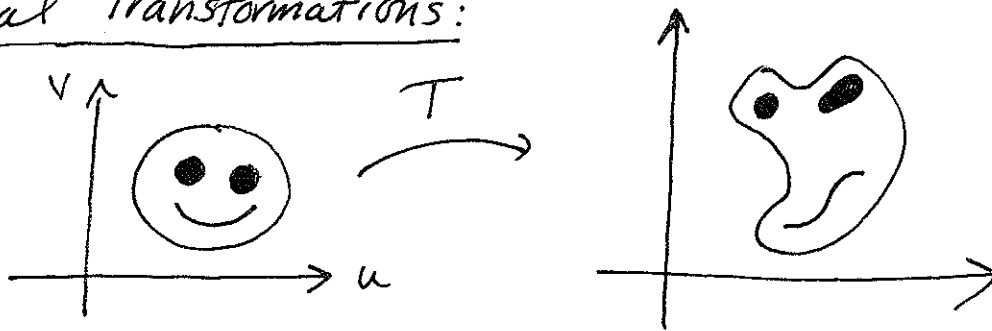
Last time: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ linear transformation asoc to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$T(u,v) = (au + bv, cu + dv)$$

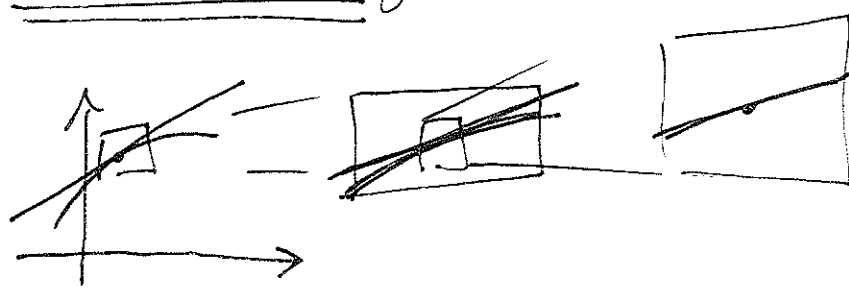


$$\iint_S f(T(u,v)) \begin{vmatrix} a & b \\ c & d \end{vmatrix} du dv = \iint_R f(x,y) dA$$

General Transformations:



[Comes back to a fundamental notion in Calculus:
Local linearity. That is, can approx f by linear fns.]



$$f(u+\Delta u) = f(u) + f'(u)\Delta u + E(\Delta u)$$

u
small

$$\approx c + a\Delta u$$

$$g(u+\Delta u, v+\Delta v) = g(u, v) + g_u(u, v)\Delta u + g_v(u, v)\Delta v + \text{Error.}$$

Say that $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is differentiable

at (u, v) if $T(u, v) = (g(u, v), h(u, v))$ and

$$T(u+\Delta u, v+\Delta v) = T(u, v) + DT(\Delta u, \Delta v) + E(\Delta u, \Delta v)$$

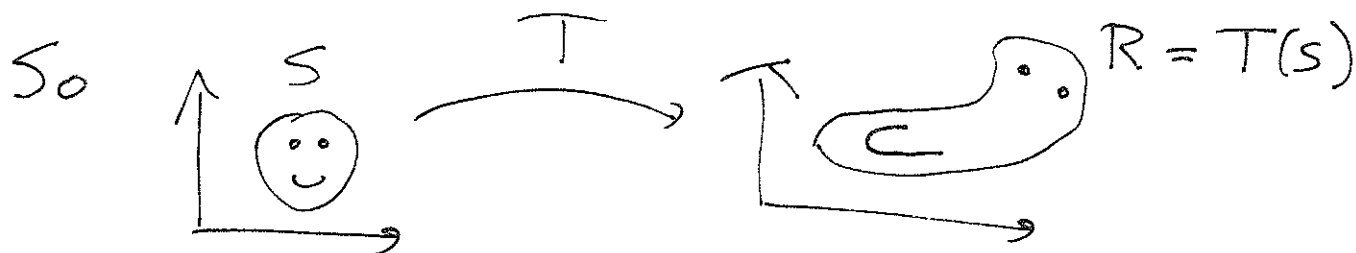
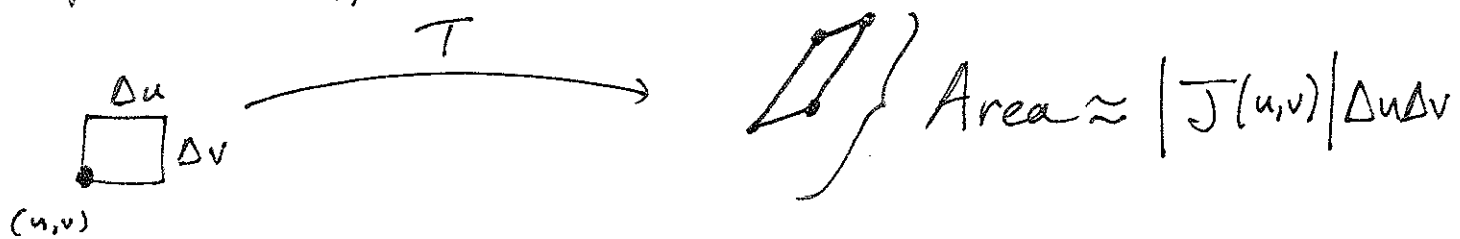
where DT is the linear trans

with matrix $J(u, v) = \underbrace{\begin{pmatrix} g_u(u, v) & g_v(u, v) \\ h_u(u, v) & h_v(u, v) \end{pmatrix}}_{\text{Jacobian matrix.}}$

and $E: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with

$$\lim_{(\Delta u, \Delta v) \rightarrow (0, 0)} \frac{|\vec{E}(\Delta u, \Delta v)|}{|(\Delta u, \Delta v)|} = 0.$$

Then if T is diff at (u, v) have



Then

$$\iint_S f(T(u, v)) |\det \mathbf{J}| \, du \, dv = \iint_R f(x, y) \, dA$$

book denotes as $\left| \frac{\partial(x, y)}{\partial(u, v)} \right|$

Ex: Polar coordinates

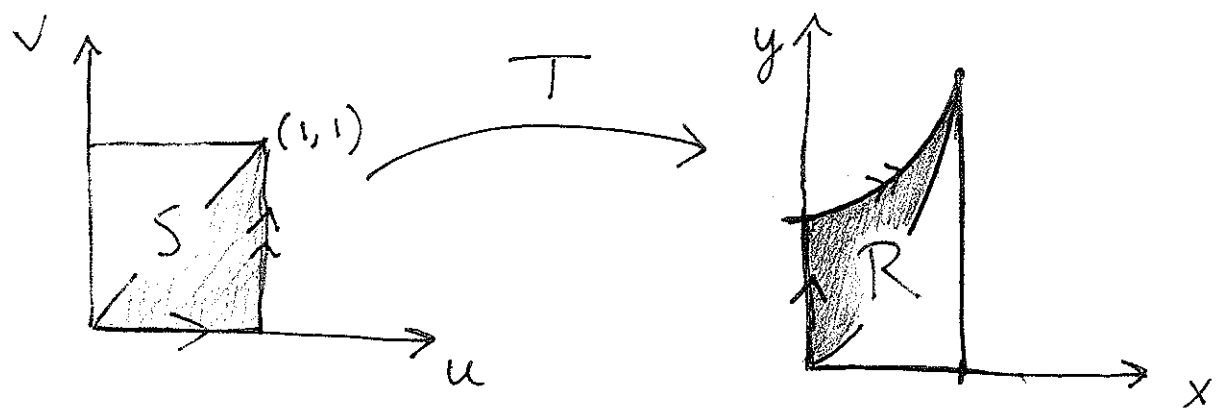
$$T(r, \theta) = (r \cos \theta, r \sin \theta)$$

$$\mathbf{J} = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$$

$$\det \text{ is } = r \cos^2 \theta + r \sin^2 \theta = r$$

So $dA = r \, dr \, d\theta$, as before.

Ex: (Worksheet) $T(u,v) = (v, u(1+v^2))$



$$J = \begin{pmatrix} 0 & 1 \\ 1+v^2 & 2uv \end{pmatrix}$$

$$|\det J| = |-1 - v^2|$$

negative because of the flip.

$$= 1+v^2$$

Compute

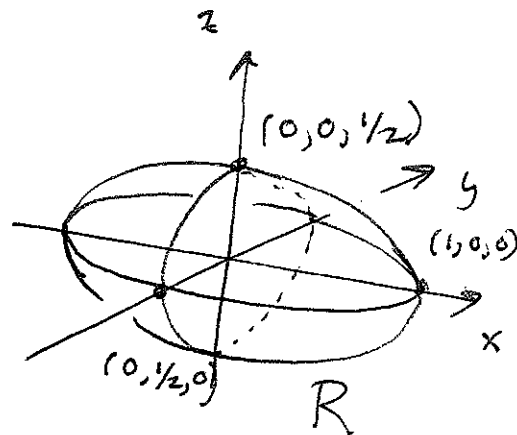
$$\iint_R x+y \, dA = \iint_S (v+u(1+v^2)) \underbrace{(1+v^2)}_{dA} \, du \, dv$$

$$= \int_0^1 \int_0^1 v+v^3+u(1+2v^2+v^4) \, dv = \int_0^1 v+v^3+\frac{1}{2}+v^2+\frac{1}{2}v^4 \, dv$$

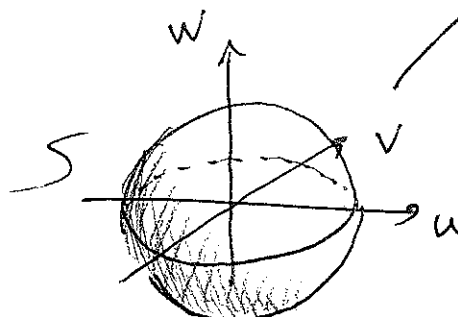
$$= \left. \frac{v^2}{2} + \frac{v^4}{4} + \frac{v}{2} + \frac{v^3}{3} + \frac{1}{10}v^5 \right|_{v=0}^1 = \frac{101}{60}$$

Changing coordinates in \mathbb{R}^3

$$R = \{ x^2 + 4y^2 + 4z^2 \leq 1 \}$$



$$\iiint_R 1 - x^2 - 4y^2 - 4z^2 \, dV$$



$$T(u, v, w) = \left(u, \frac{v}{2}, \frac{w}{2} \right)$$

Since.

$$1 \geq x^2 + 4y^2 + 4z^2 = u^2 + v^2 + w^2$$

T is linear, with matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} = J$

Changes volume by $\det(J) = 1/4$.

So

$$\iiint_R 1 - x^2 - 4y^2 - 4z^2 \, dV$$

$$= \iiint_S (1 - u^2 - v^2 - w^2) \frac{1}{4} \, du \, dv \, dw$$

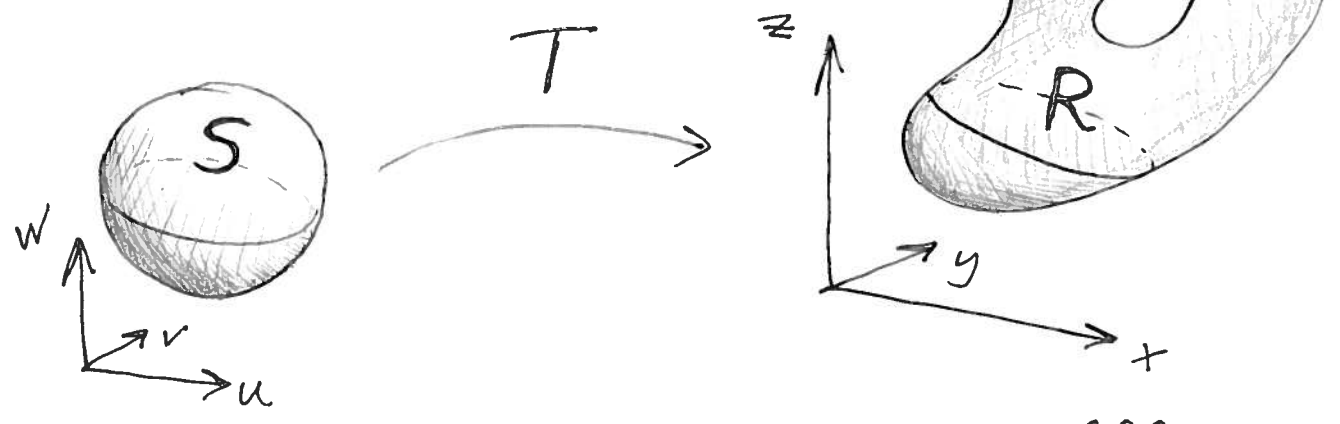
Once more, with feeling.

$$= \int_0^1 \int_0^\pi \int_0^{2\pi} (1-\rho^2) \frac{1}{4} \rho^2 \sin \phi \, d\theta \, d\phi \, d\rho$$

$$= \int_0^1 -\frac{\pi}{2} (1-\rho^2) \rho^2 \cos \phi \Big|_{\phi=0}^{\phi=\pi} d\rho = \int_0^1 \pi (\rho^2 - \rho^4) d\rho$$

$$= \pi \left(\frac{\rho^3}{3} - \frac{\rho^5}{5} \right) \Big|_{\rho=0}^{\rho=1} = \pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{2\pi}{15}$$

In general, for $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$



$$\iiint_S f(T(u,v,w)) |\det J| \, du \, dv \, dw = \iiint f(x,y,z) \, dV$$

where

$$J = \begin{pmatrix} \frac{\partial T_1}{\partial u} & \frac{\partial T_1}{\partial v} & \frac{\partial T_1}{\partial w} \\ \frac{\partial T_2}{\partial u} & \frac{\partial T_2}{\partial v} & \frac{\partial T_2}{\partial w} \\ \frac{\partial T_3}{\partial u} & \frac{\partial T_3}{\partial v} & \frac{\partial T_3}{\partial w} \end{pmatrix} \text{ if } T = (T_1, T_2, T_3).$$