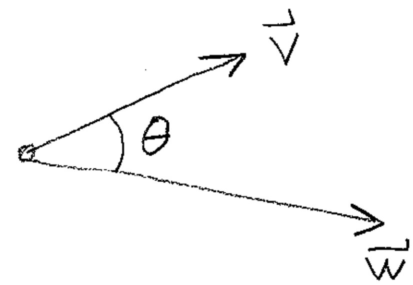
Lecture 3: Dot product (12.3) and lines and planes in $\mathbb{R}^3$ (12.5)

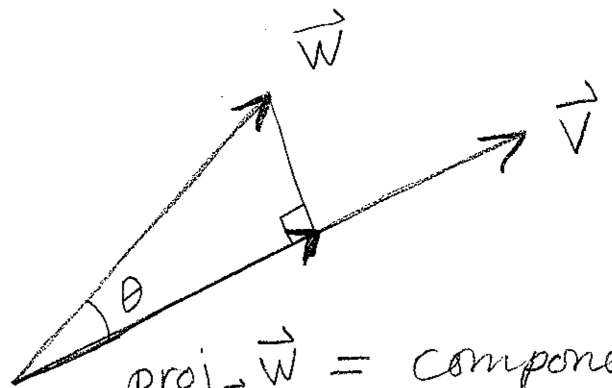
<u>Last time:</u> $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3)$

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

<u>Key:</u> $\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta$



<u>Projection:</u>



$$\text{proj}_{\vec{v}} \vec{w} = \text{component of } \vec{w} \text{ along } \vec{v}$$

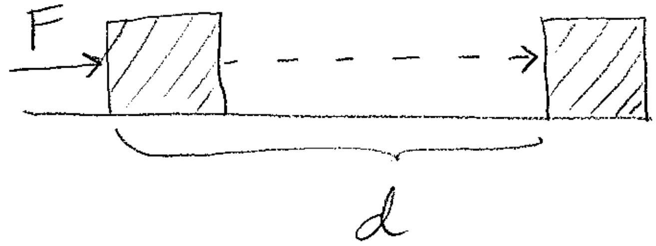
$= \text{scalar mult of } \vec{v}$ closest to $\vec{w}$.

<u>Note:</u> $|\text{proj}_{\vec{v}} \vec{w}| = |\vec{w}| \cos\theta$

<u>So:</u> $\text{proj}_{\vec{v}} \vec{w} = |\vec{w}| \cos\theta \left(\text{unit vector pointing in same dir as } \vec{v} = \dfrac{\vec{v}}{|\vec{v}|} \right)$

$$= \frac{|\vec{v}||\vec{w}| \cos\theta}{|\vec{v}|^2} \vec{v} = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}|^2} \vec{v}$$

Work = (force) × (distance)

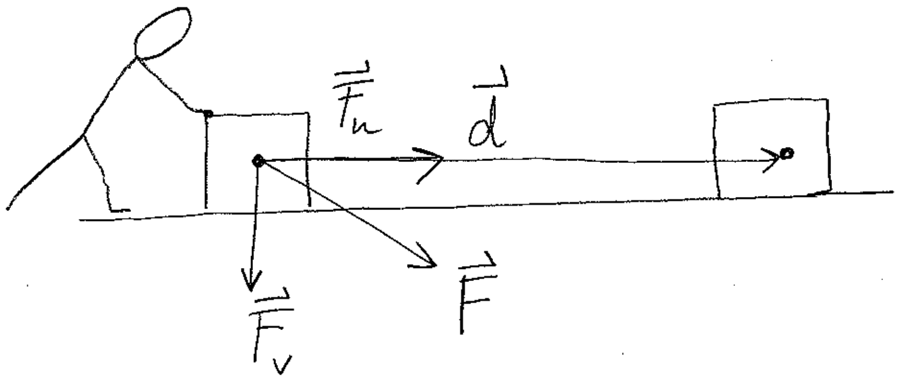


$$W = |\vec{F}_h||\vec{d}|$$

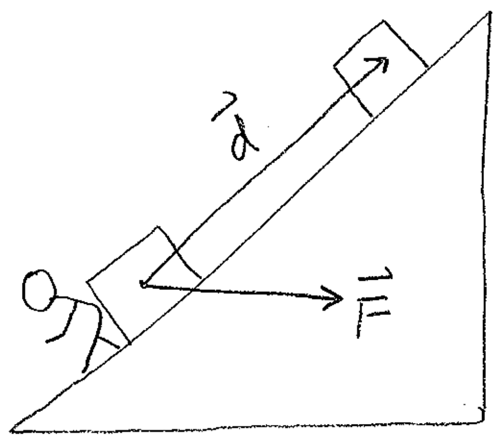
$$= |\mathrm{proj}_{\vec{d}}\vec{F}||\vec{d}|$$

$$= \left|\frac{\vec{F}\cdot\vec{d}}{|\vec{d}|^2}\vec{d}\right||\vec{d}|$$

$$= \vec{F}\cdot\vec{d}$$



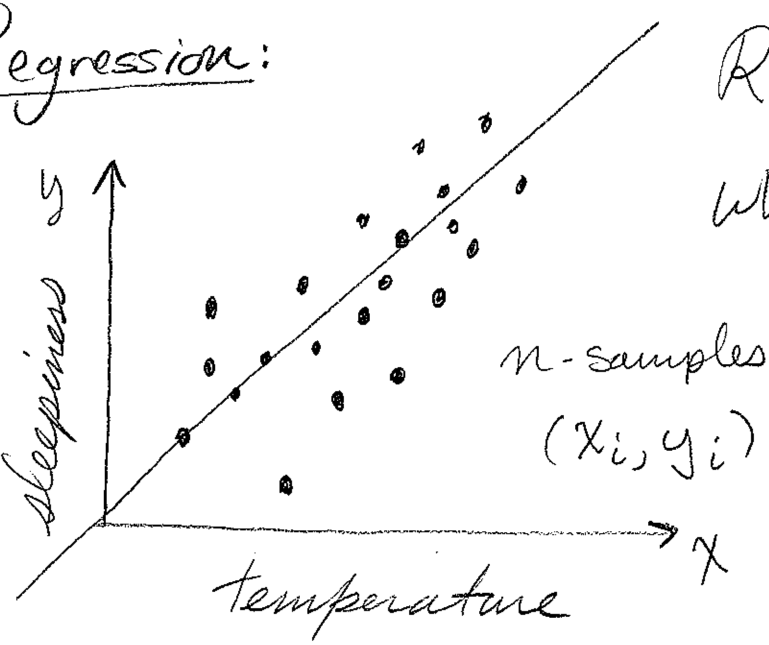
$$\boxed{W = \vec{F}\cdot\vec{d}}$$



Regression:

Roughly $y = cx$.

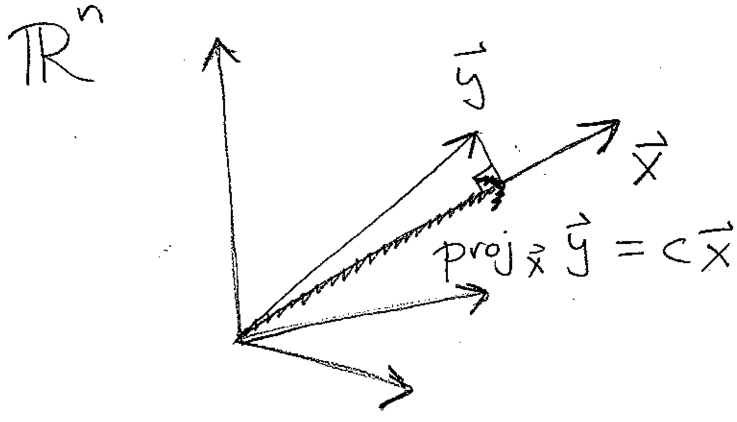
What is c?



$\vec{x} = (x_1, x_2, x_3, \ldots, x_n)$ $\quad$ $\vec{y} = (y_1, y_2, y_3, \ldots, y_n)$

If $y_i = c x_i$ for each i, then

$$\vec{y} = c\vec{x}$$

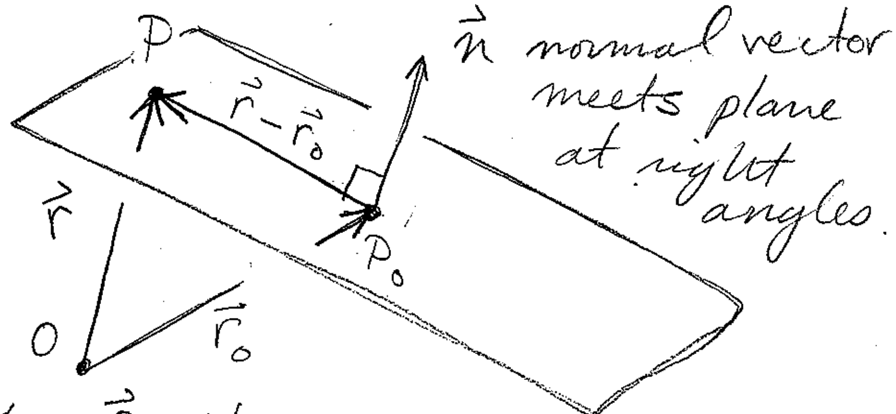


So "best fit" is

$$c = \frac{\vec{x}\cdot\vec{y}}{|\vec{x}|^2}$$

which minimizes $|\vec{y} - c\vec{x}|$ [In general model has more parameters, and proj is onto a plane or similar. Cf. Math 415.]

Planes in $\mathbb{R}^3$:



Take a point $P_0 = (x_0, y_0, z_0)$ on the plane, with $\vec{r}_0$ its pos. vector.

Test if $P = (x, y, z)$ is in the plane:

$\vec{n}$ and $\vec{r} - \vec{r}_0$ are perpendicular (orthogonal),

i.e. $\vec{n}\cdot(\vec{r} - \vec{r}_0) = 0$

If $\vec{n} = (a, b, c)$, have $P = (x, y, z)$ in the plane when

$$0 = (a, b, c) \cdot (x - x_0, y - y_0, z - z_0)$$

$$= a(x - x_0) + b(y - y_0) + c(z - z_0)$$

$$= ax + by + cz + d$$

with $d = -(ax_0 + by_0 + cz_0)$

Conversely,

$$ax + by + cz + d = 0$$

defines a plane [unless $a = b = c = 0$.]

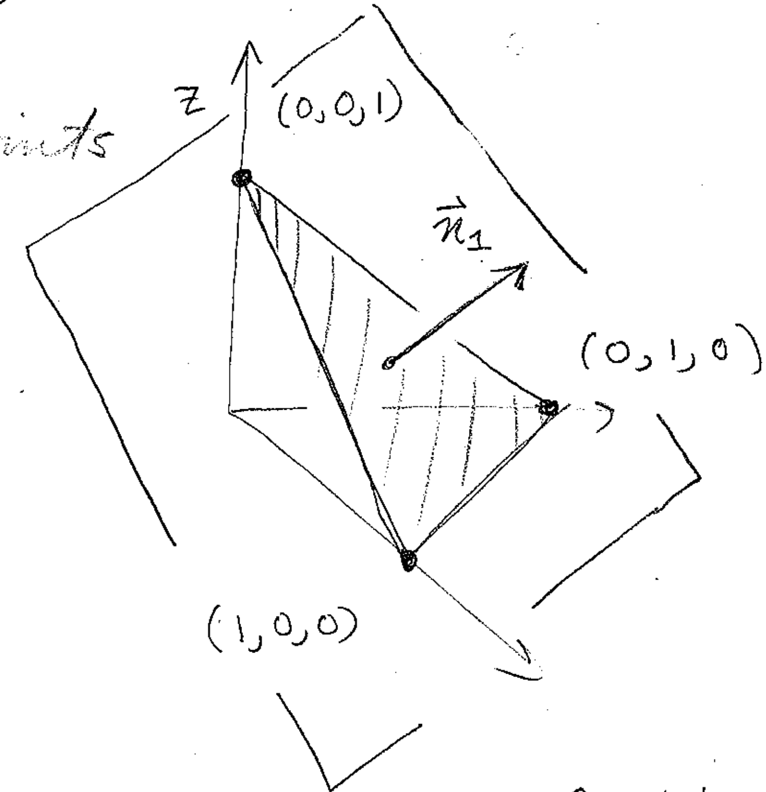
[Normal vectors will be a key concept later in the course. Can be used to solve geometric problems about planes]

Ex: P_1 = plane given by $x+y+z-1=0$

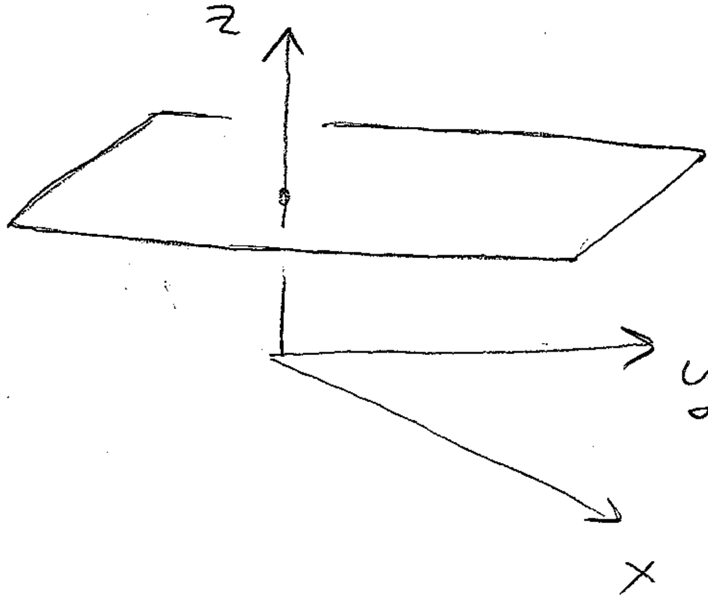
Plane is det by 3-points

Normal vector

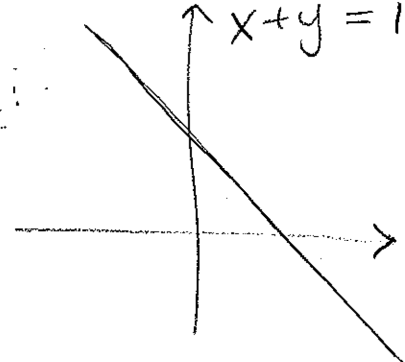
$\vec{n}_1 = (1,1,1)$



$P_2 = \{ z - 1 \}$



Compare:



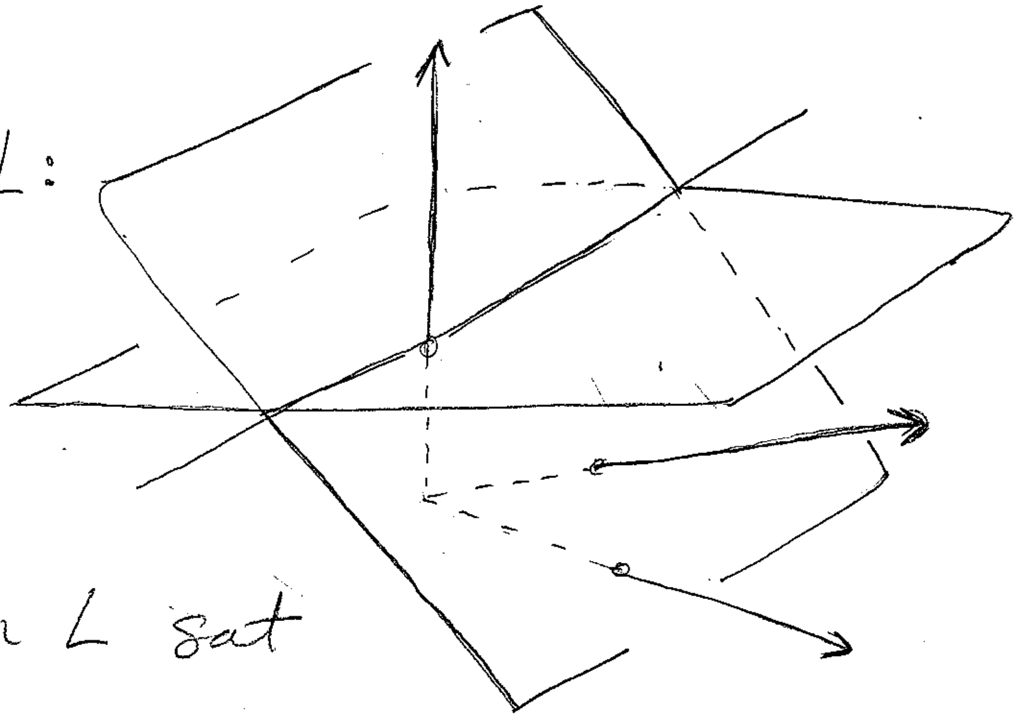
$\vec{n}_2 = (0,0,1)$

Q1: What is the intersection of P_1 and P_2?

Q2: What is the angle between P_1 and P_2?

A1:

A line L:



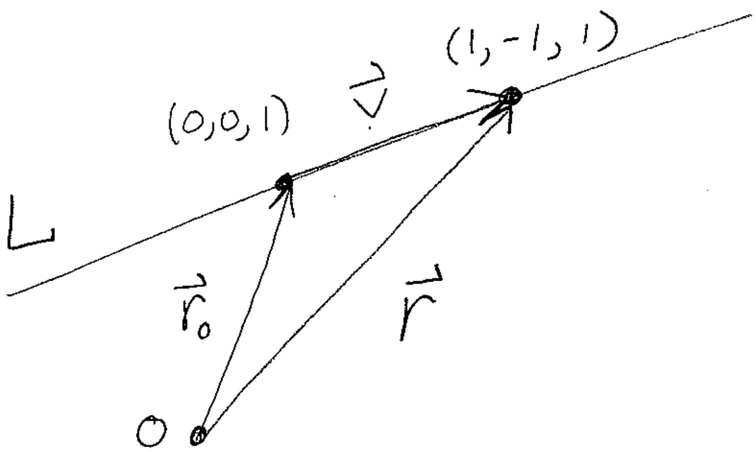
points on L sat

$$x+y+z=1 \quad \text{and} \quad z=1$$

Two easy solutions:

$\vec{r}_0 = (0,0,1)$

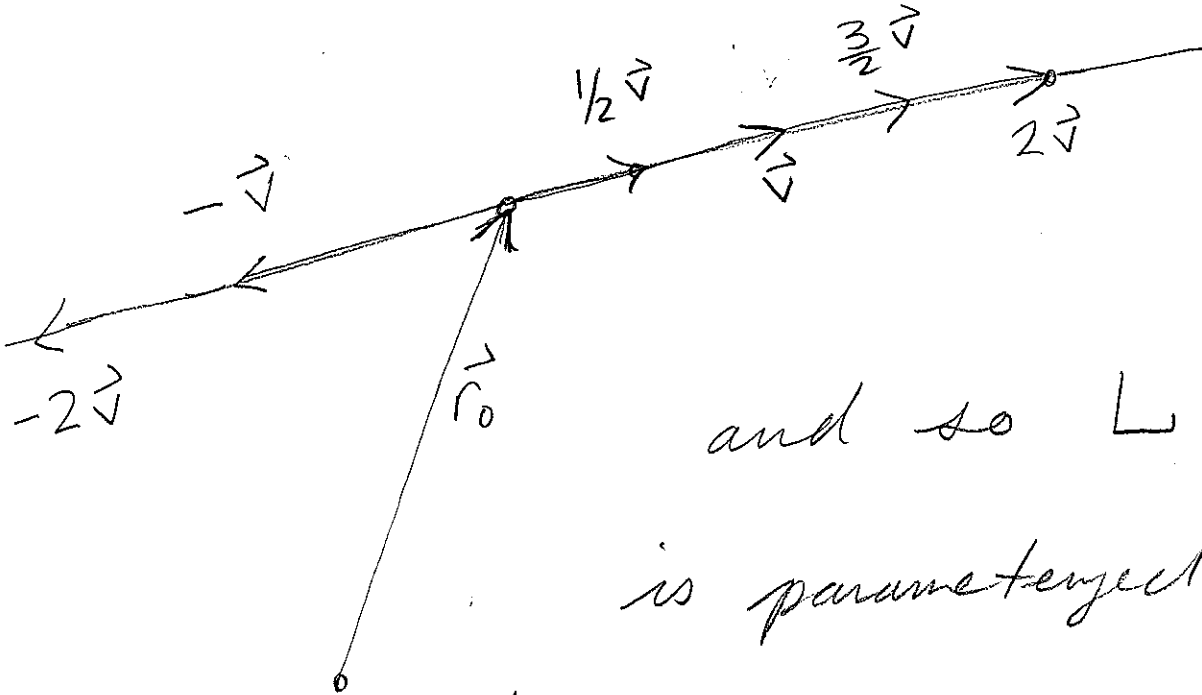
$\vec{r} = (1,-1,1)$



$$\vec{v} = \vec{r} - \vec{r}_0 = (1,-1,0)$$

Every pt on L has the form

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}$$



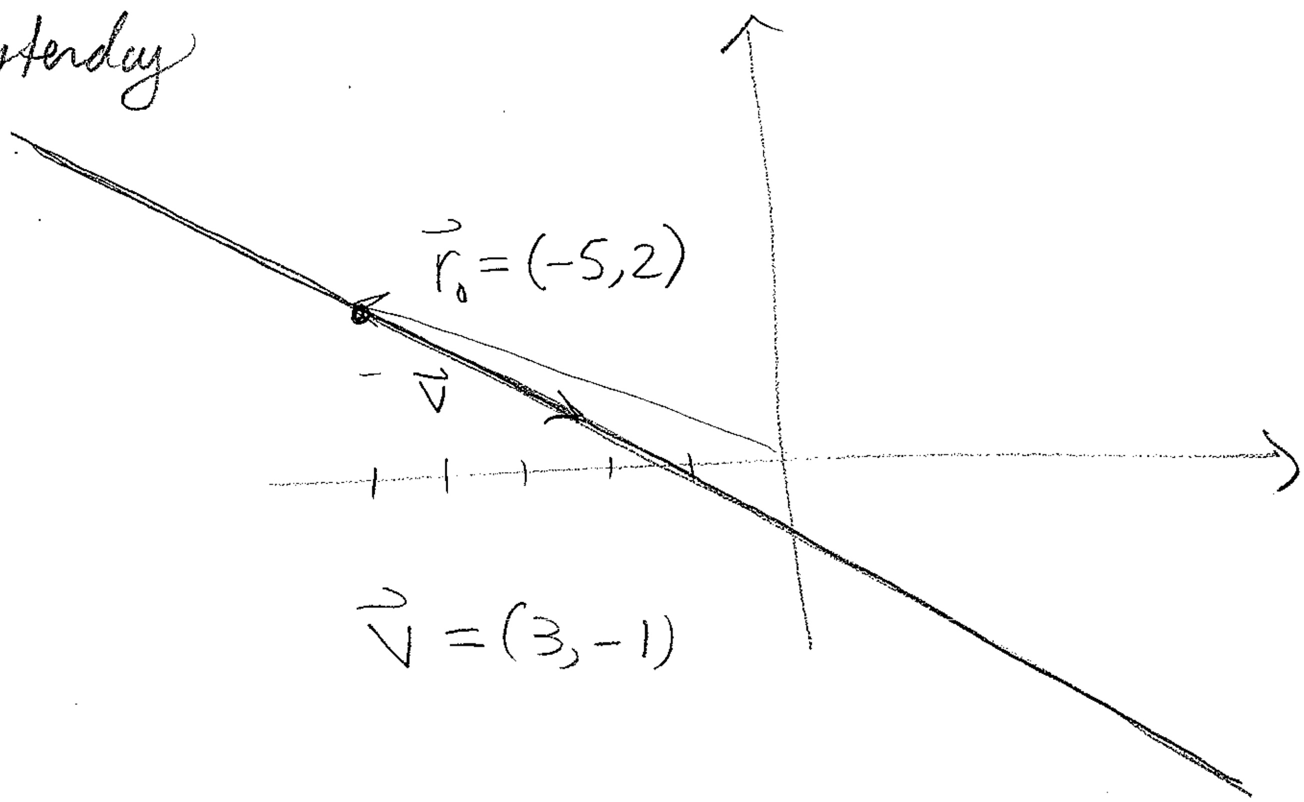
and so L

is parametrized

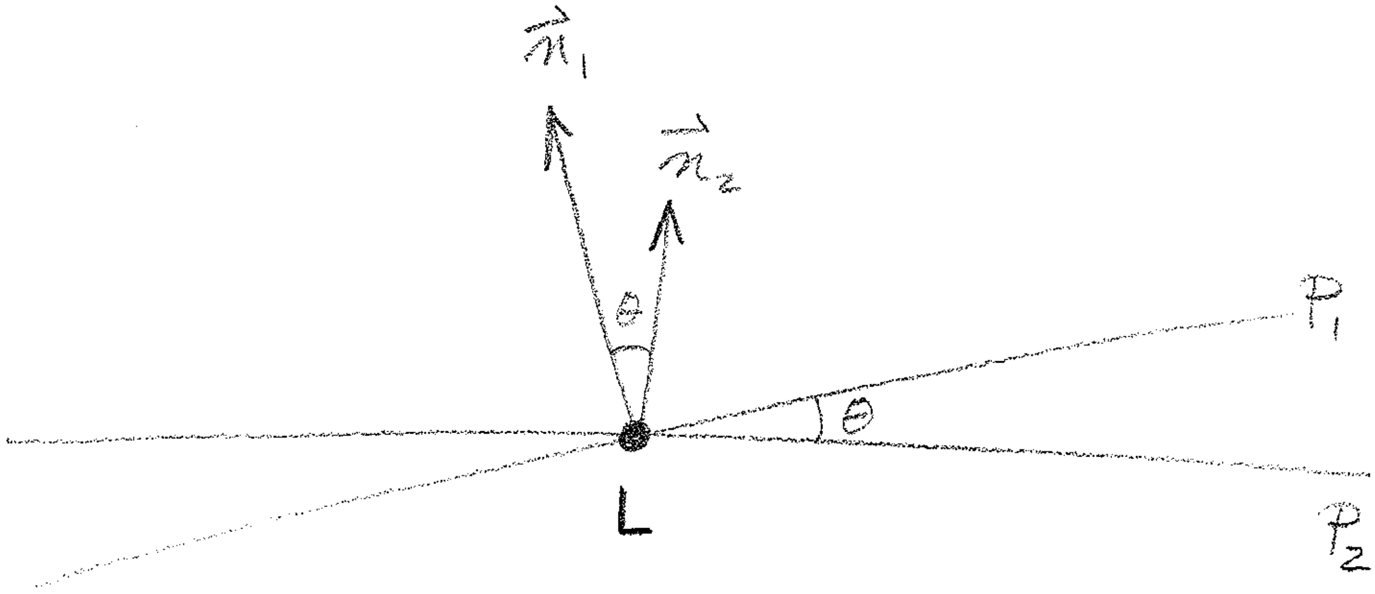
by

$$(0,0,1)+t(1,-1,0) = (t,-t,1)$$

This is just like worksheet 3(e) from yesterday



A2: Angle between planes is same as angle between normals



$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{1}{\sqrt{3}\cdot 1} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta \simeq 54.7^\circ$$