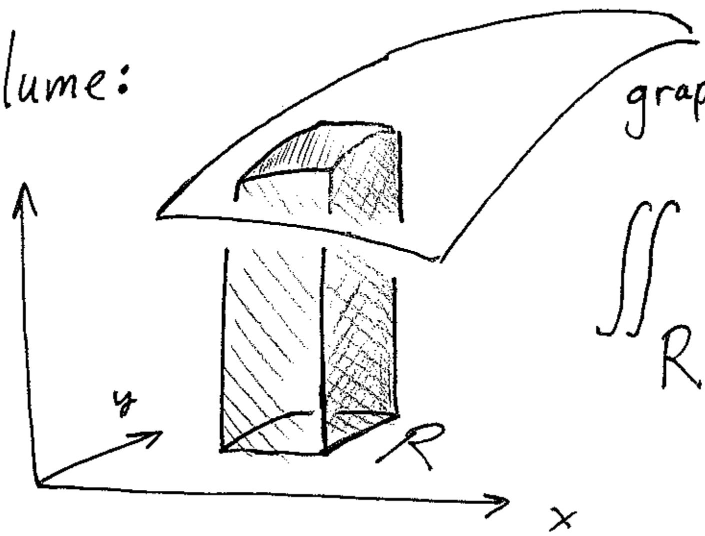
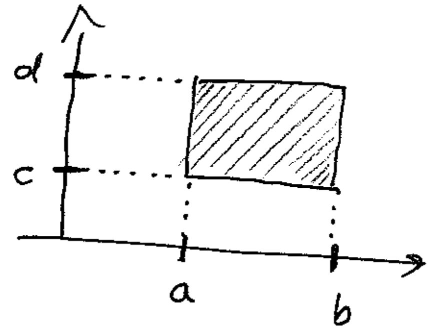
Lecture 25: Integrating over more complicated regions (15.2 – 15.4)

<u>Last time</u>: $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad$ R = rectange in $\mathbb{R}^2$.

Volume:

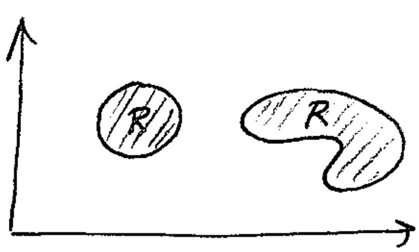


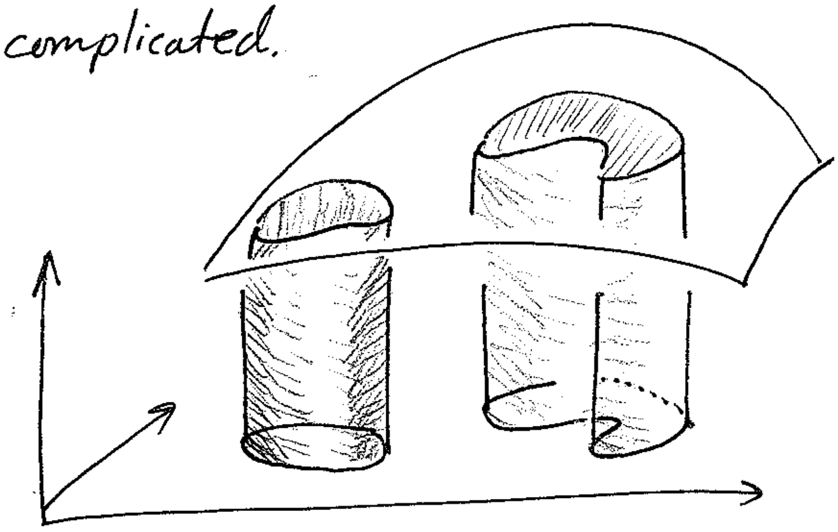


$$\iint_R f(x,y)\, dA = \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy$$

[Other interpretions: averages, total mass, center of mass, ...]

<u>Today</u>: When R is more complicated.

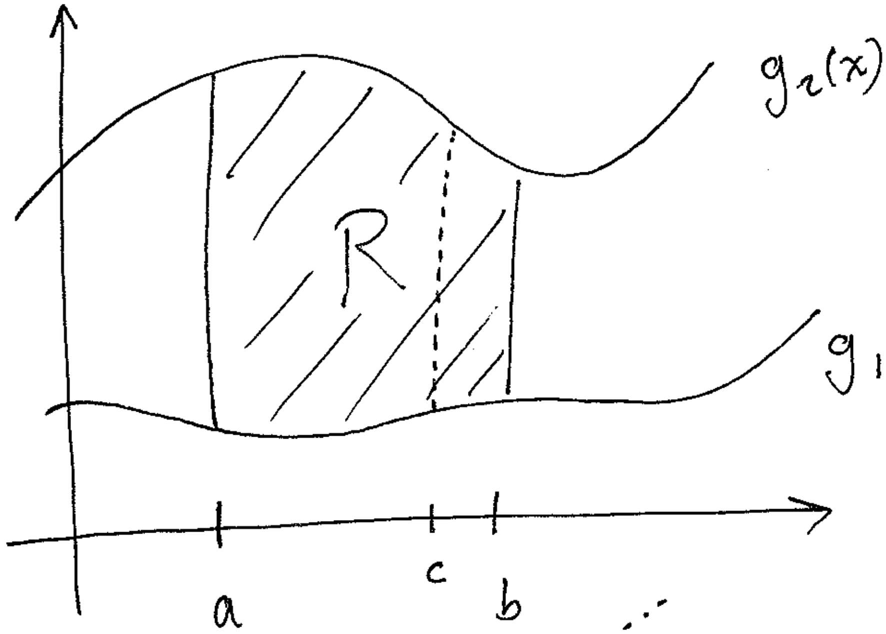




<u>Q</u>: How to compute

$$\text{Volume} = \iint_R f(x,y)\, dA \, ?$$

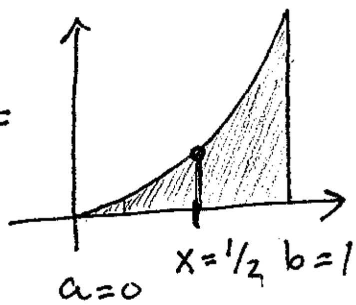
Suppose R has the form:



$$R = \left\{ \begin{array}{l} a \le x \le b \ \underline{\text{and}} \\ g_1(x) \le y \le g_2(x) \end{array} \right\}$$

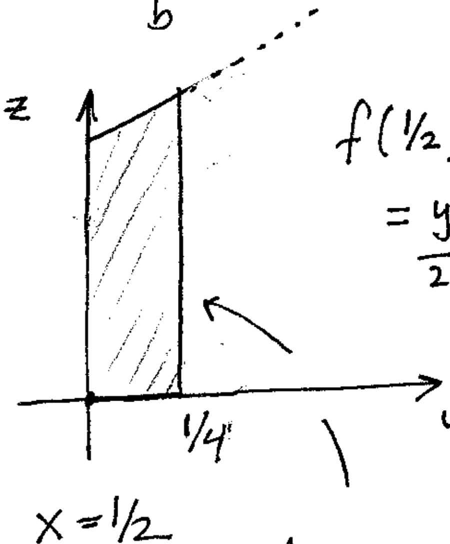
<u>Idea</u>: Slice along lines $x = c$.

<u>Ex</u>: $R =$



$g_2 = x^2$

$g_1 = 0$



$f(1/2, y) = \frac{y}{2} + 1$

Area $= A(1/2)$

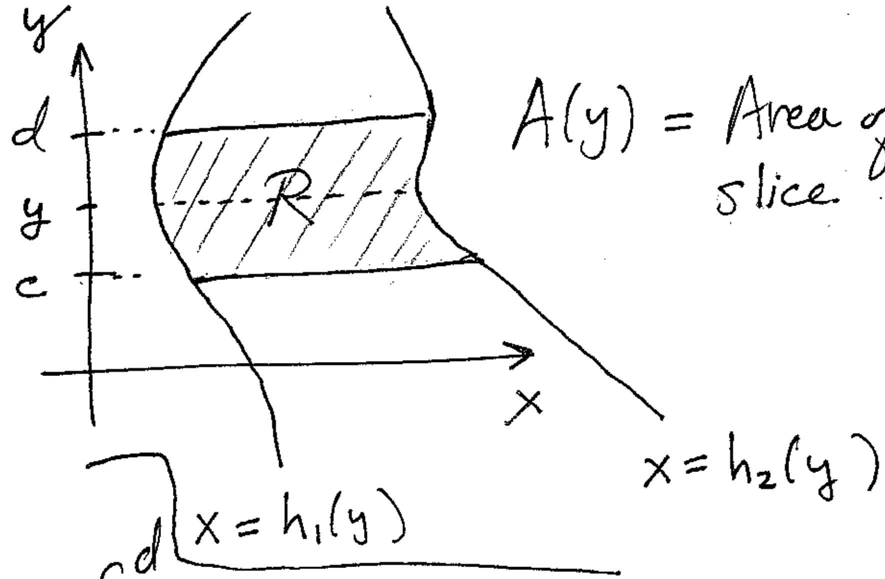
$$\iint_R \underbrace{xy + 1}_{f(x,y)} \, dA = \int_0^1 A(x)\, dx$$

$$= \int_0^1 \left(\int_0^{x^2} xy + 1 \, dy \right) dx = \int_0^1 \left(\frac{xy^2}{2} + y \Big|_{y=0}^{y=x^2} \right) dx$$

$$= \int_0^1 \frac{1}{2} x (x^2)^2 + x^2 \, dx = \int_0^1 \frac{1}{2} x^5 + x^2 \, dx$$

$$= \frac{1}{12} x^6 + \frac{x^3}{3} \Big|_{x=0}^{1} = \frac{1}{12} + \frac{1}{3} = \frac{5}{12}$$

Similar case:

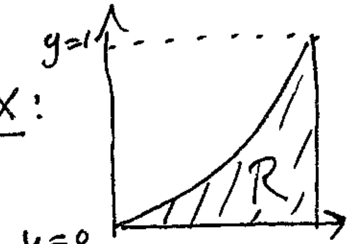


$A(y)$ = Area of slice

$$\iint_R f(x,y)\, dA = \int_c^d A(y)\, dy$$

$$= \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\, dx \right) dy$$

Ex:



$h_1(y) = \sqrt{y}$
$h_2(y) = 1$

$$\iint_R xy + 1\, dA = \int_0^1 A(y)\, dy = \int_0^1 \left(\int_{\sqrt{y}}^{1} xy+1\, dx \right) dy$$

$$= \int_0^1 \left(\frac{1}{2}x^2 y + x \Big|_{x=\sqrt{y}}^{x=1} \right) dy = \int_0^1 \left(\frac{1}{2}y+1\right) - \left(\frac{1}{2}y^2 + \sqrt{y}\right) dy$$

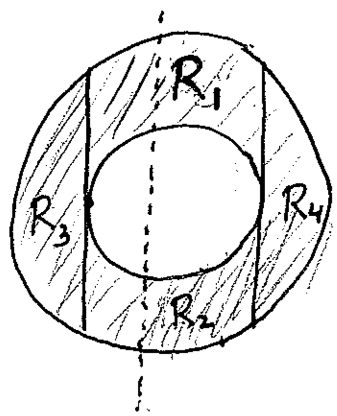
$$= \int_0^1 -\frac{1}{2}y^2 + \frac{1}{2}y - \sqrt{y} + 1\, dy = -\frac{y^3}{6} + \frac{y^2}{4} - \frac{2}{3}y^{3/2} + y \Big|_{y=0}^{1}$$

$$= -\frac{1}{6} + \frac{1}{4} - \frac{2}{3} + 1 = \frac{-2+3-8+12}{12} = \frac{5}{12} \checkmark$$

[Book calls these two kinds of regions type I and type II. Some regions are both, in which case it can be easier to do things one way or the other.]

General region:

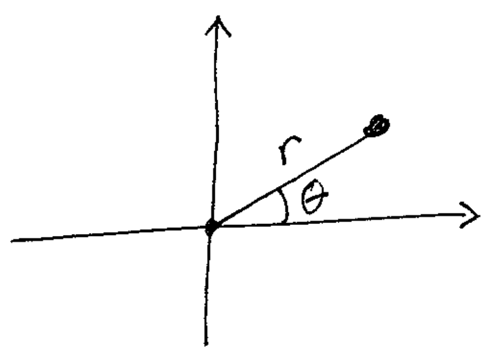
(A) Cut into simple pieces.



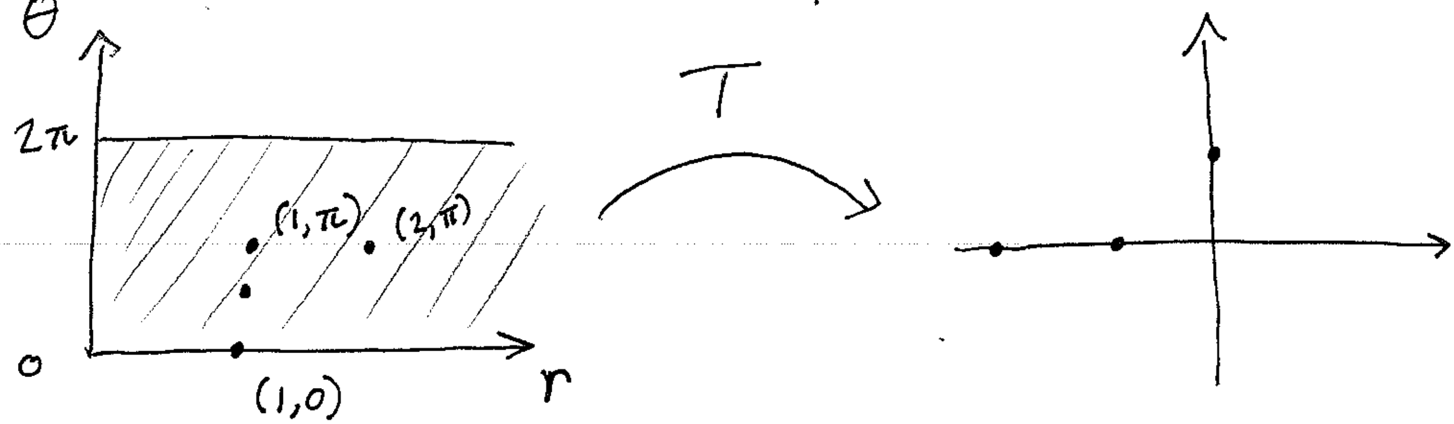
$$\iint_R f\,dA = \sum_{i=1}^{4} \iint_{R_i} f\,dA$$

(B) Change Coordinates, so R becomes easier to describe.

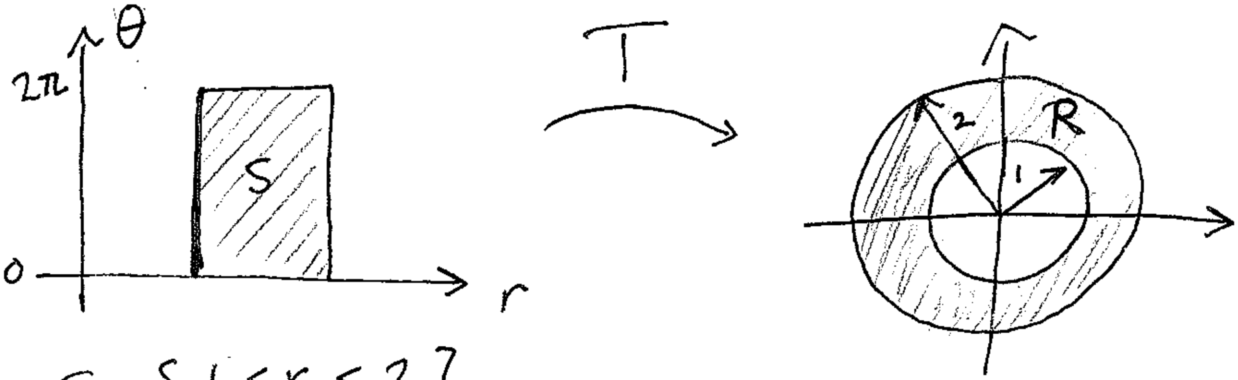
Polar Coordinates:



$$x = r\cos\theta$$
$$y = r\sin\theta$$



$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \qquad T(r,\theta) = (r\cos\theta,\ r\sin\theta)$$



$S = \left\{ \begin{matrix} 1 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{matrix} \right\}$

$$\boxed{R = T(S).}$$

<u>Goal</u>: Relate $\iint_R f\, dA$ to an integral over S.

<u>First try</u>:

$$\iint_R 1\, dA \overset{\text{Guess!}}{=} \iint_S 1\, dA = \int_0^{2\pi} \int_1^2 1\, dr\, d\theta$$

$$= \int_0^{2\pi} r \Big|_{r=1}^{2} d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi.$$

<u>Is this right</u>?

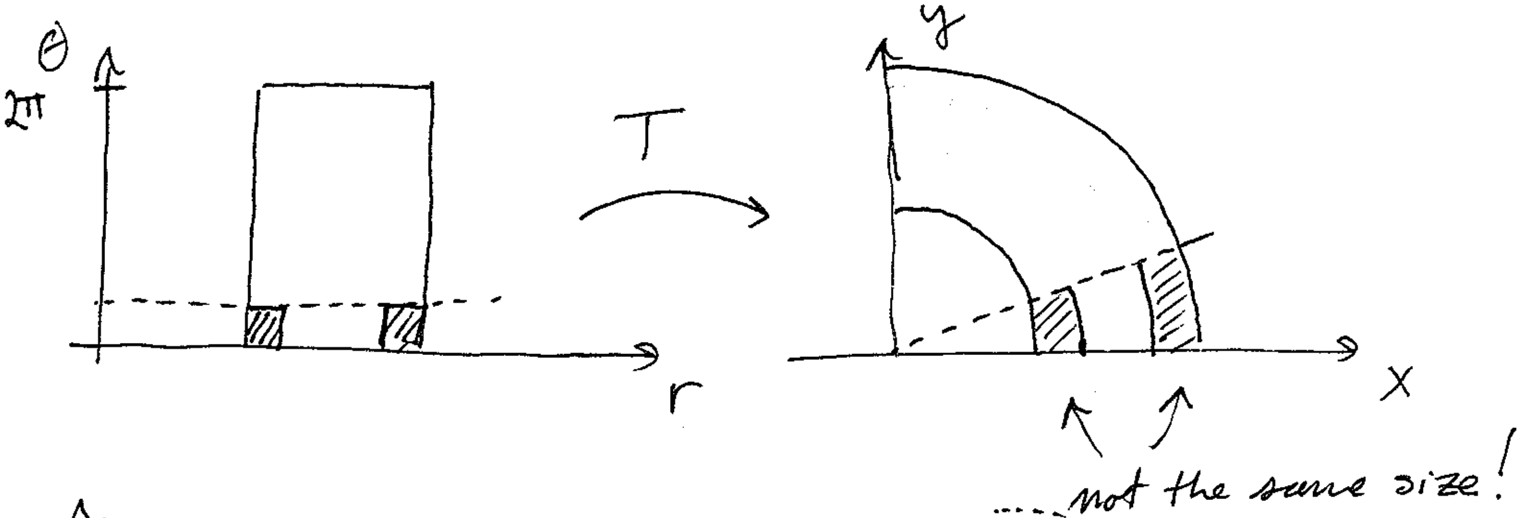
$$\iint_R 1\, dA = \text{Area}(R) = \text{Area}\begin{pmatrix}\text{disc of} \\ \text{rad } 2\end{pmatrix} - \text{Area}\begin{pmatrix}\text{disc of} \\ \text{rad } 1\end{pmatrix}$$

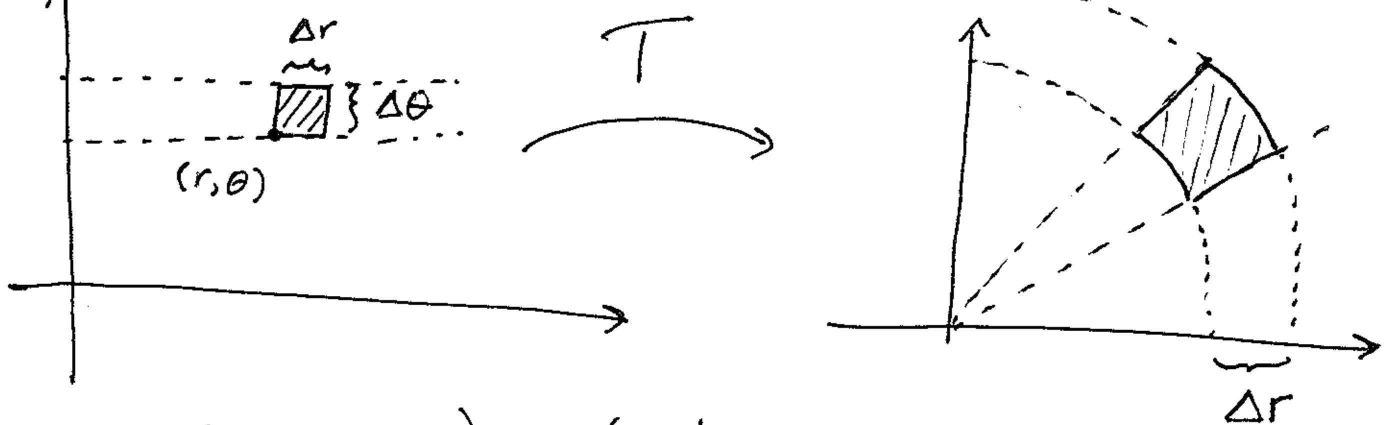
$$= \pi 2^2 - \pi 1^2 = 3\pi.$$

So this didn't work...

Source of problem: T distorts area

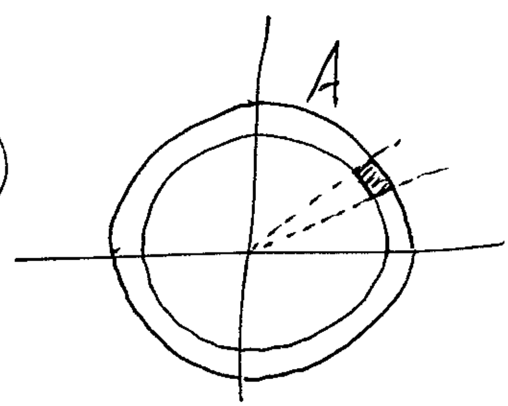
(in a nonuniform way).





$$\text{Area}\big(T(\square)\big) = \begin{pmatrix}\text{portion} \\ T(\square) \text{ is} \\ \text{of the annulus } A\end{pmatrix} \cdot \begin{pmatrix}\text{Area of} \\ A\end{pmatrix}$$

$$= \frac{\Delta\theta}{2\pi}\Big(\underbrace{\pi(r+\Delta r)^2 - \pi r^2}_{\pi(r^2+2r\Delta r+\Delta r^2 - r^2)}\Big)$$



$$= r\,\Delta r\,\Delta\theta + \tfrac{1}{2}\Delta r^2\,\Delta\theta$$

$$\approx r\,\Delta r\,\Delta\theta \quad \text{if } \Delta r \text{ is small.}$$

$$\text{Area}(S) = \sum_{\substack{\text{sub rectangles} \\ \text{of } R}} \text{Area}\big(T(\square)\big) \approx \sum r\,\Delta r\,\Delta\theta$$

$\approx \iint_S r\,dr\,d\theta$. So, should have

$$\text{Area}(S) = \iint_S r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 r\,dr\,d\theta = \int_0^{2\pi} \frac{r^2}{2}\Big|_{r=1}^{2}\,d\theta$$

$$= \int_0^{2\pi} \frac{2^2}{2} - \frac{1^2}{2}\,d\theta = \int_0^{2\pi} \frac{3}{2}\,d\theta$$

$= \frac{3}{2}\cdot 2\pi = 3\pi$, which matches our geometric answer!

<u>Summary:</u> When using polar coordinates

$dA = r\,dr\,d\theta$ <u>not</u> $dr\,d\theta$.