Lecture 23: Conservative vector fields II (16.3)

Last time: $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ is conservative if $\vec{F} = \nabla f$ for some $f: \mathbb{R}^n \to \mathbb{R}$.

Note: HW for today included w/ last assig.

Thm A: $\vec{F}$ a vector field on an open connected region D in $\mathbb{R}^2$. $\vec{F}$ is conservative if and only if $\int_C \vec{F} \cdot d\vec{r}$ is path independent.

Thm B: $\vec{F} = (P, Q)$ on simply connected open D in $\mathbb{R}^2$. $\vec{F}$ is conserv. if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D.

Thm A also works for $\mathbb{R}^n$.

Thm B has analogs, but more complicated. (16.3 #27)

[Note to self: right cond is $H^1(D;\mathbb{R}) = 0$.]

Do physics first!

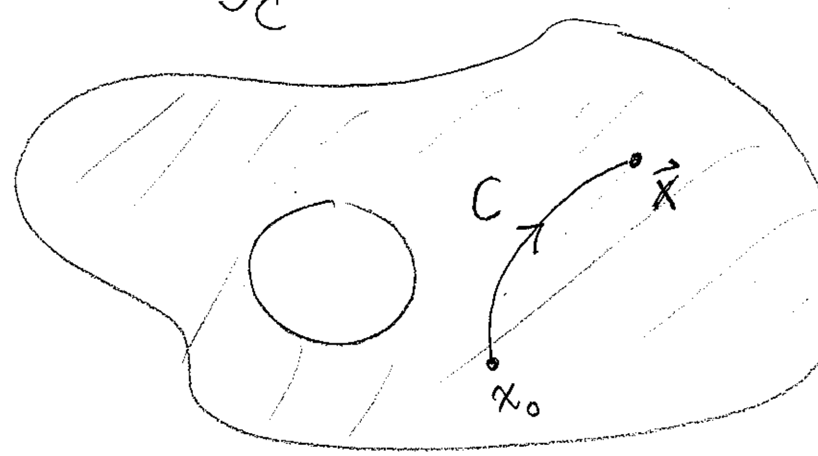
Reason for Thm A: Suppose $\int_C \vec{F} \cdot d\vec{r}$ is path-indep.

Pick $\vec{x}_0$ in D

Define

$f: D \to \mathbb{R}$

by



$$f(\vec{x}) = \int_C \vec{F} \cdot d\vec{r} \quad \text{for any path } C \text{ from } \vec{x}_0 \text{ to } \vec{x}.$$

[This is an odd sort of fn, but it is a way of assoc. a number to each input...]

<u>Note</u>: $f(\vec{x}_0) = 0$ and so the F.T.L.I says:

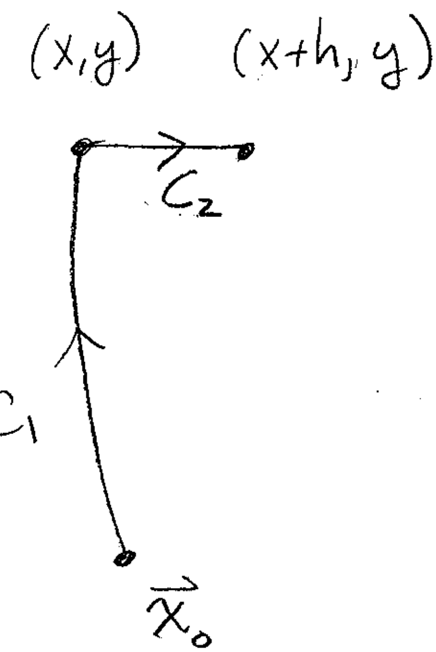
$$\int_C \nabla f \cdot d\vec{r} = f(\vec{x}) - f(\vec{x}_0) = f(\vec{x})$$

<u>Point</u>: $\nabla f = \vec{F}$. For instance, let's compute

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h \to 0} \frac{1}{h} \int_{C_2} \vec{F} \cdot dr$$

$$f(x,y) = \int_{C_1} \vec{F} \cdot d\vec{r}$$

$$f(x+h,y) = \int_{C_1 + C_2} \vec{F} \cdot d\vec{r}$$



Now if $\vec{F} = (P, Q)$ then as $\vec{T} = (1, 0)$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot \vec{T}\, ds = \int_{C_2} P\, ds$$

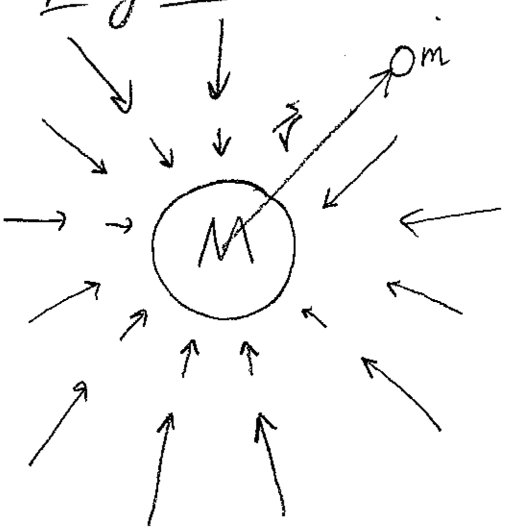
So

$$\frac{\partial f}{\partial x}(x,y) = \lim_{h \to 0} \underbrace{\frac{1}{h} \int_{C_2} P\, ds}_{\text{Average of } P \text{ on } C_2} = P(x,y).$$

Averaging over shorter and shorter curves.

Likewise $\dfrac{\partial f}{\partial y} = Q$ and so $\nabla f = \vec{F}$, i.e. $\vec{F}$ is conserv.

<u>Physical motivation</u>: Force $\vec{F} = -\dfrac{MmG}{|\vec{r}|^3}\vec{r}$



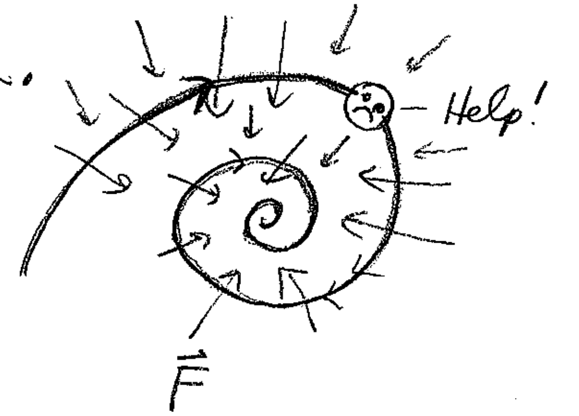
$V = -\dfrac{MmG}{|\vec{r}|}$ Potential Function

<u>Note</u>: $\vec{F} = -\nabla V$

V is larger the farther away the smaller mass is.

$\vec{F}$ points in the direction of fastest <u>decrease</u> in V.

More generally, suppose an object's path is $\vec{r}: \mathbb{R} \to \mathbb{R}^2$ acted on by a conservative force, i.e. $\vec{F} = -V$ for some $V: \mathbb{R}^2 \to \mathbb{R}$.



Newton's Law: $F = ma$ or

$$\vec{F}(\vec{r}(t)) = m\,\vec{r}\,''(t)$$

Total Energy:

$$E(t) = \begin{pmatrix}\text{Kinetic}\\ \text{energy}\end{pmatrix} + \begin{pmatrix}\text{potential}\\ \text{energy}\end{pmatrix}$$

$$= \frac{1}{2} m\, |\vec{r}\,'(t)|^2 + V(\vec{r}(t))$$

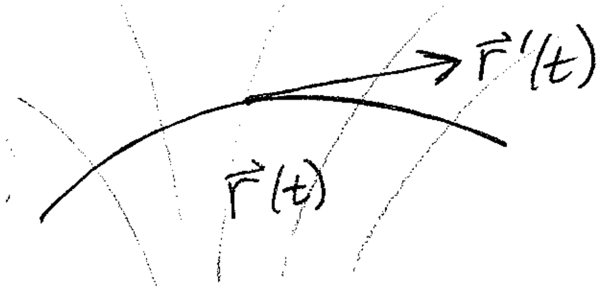
<u>Conservation of Energy</u>: $E(t)$ is const, indep of t.

<u>Reason</u>: Let's compute $E'(t)$. If $\vec{r} = (r_1, r_2)$

$$\frac{d}{dt}\left(\frac{1}{2} m\, |\vec{r}\,'(t)|^2\right) = \frac{1}{2} m \cdot \frac{d}{dt}\left((r_1'(t))^2 + (r_2'(t))^2\right)$$

$$= \frac{1}{2} m \left(2 r_1'(t) r_1''(t) + 2 r_2'(t) r_2''(t)\right)$$

$$= m\,\vec{r}\,''(t) \cdot \vec{r}\,'(t)$$



and

$$\frac{d}{dt} V(\vec{r}(t)) = \begin{array}{c}\text{rate } V \\ \text{changes} \\ \text{as a fn of } t\end{array} = \begin{array}{c}\text{rate } V \\ \text{changes as} \\ \text{we move in} \\ \text{dir } \vec{r}\,'(t)\end{array}$$

$$= D_{\vec{r}\,'(t)} V(\vec{r}(t)) = \nabla V(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

So

$$E'(t) = m\,\vec{r}\,''(t) \cdot \vec{r}\,'(t) + \nabla V(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$\updownarrow$ Newton!

$$= \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) - \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)$$

$$= 0.$$

Hence energy is conserved.