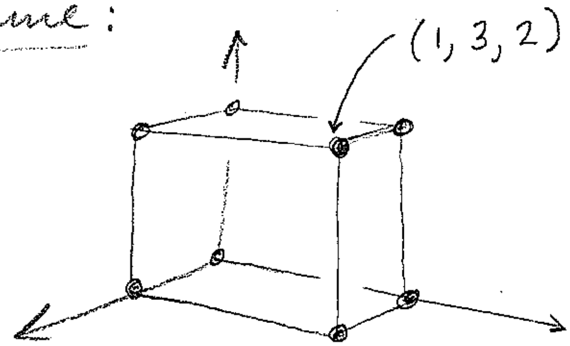
Lecture 2: Vectors (Section 12.2)

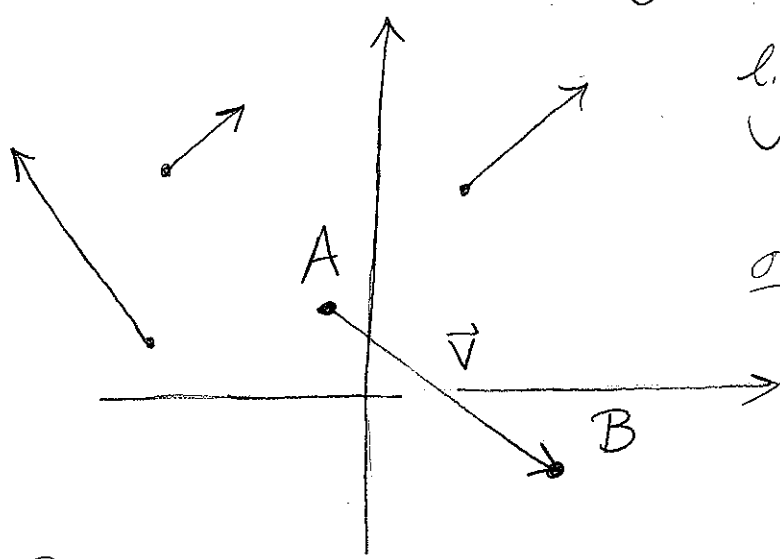
Last time:

$\mathbb{R}^3$:



$\mathbb{R}^n$ = tuples $(x_1, x_2, \ldots, x_n)$ of real #s.

Vectors in $\mathbb{R}^2$: arrow where both direction and length are important

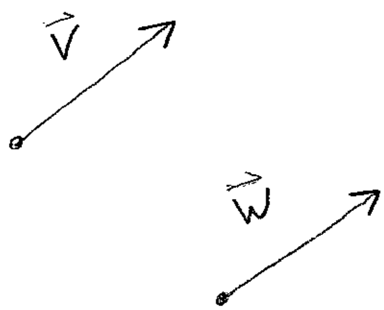


e.g. use to denote windspeed and direction

or record relative positions of two points.

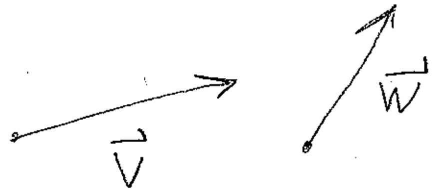
Denoted $\vec{v}$ or $\vec{w}$ or similar. [bold used in books]

$\vec{v}$ and $\vec{w}$ are regarded as equal if they are the same up to translation.

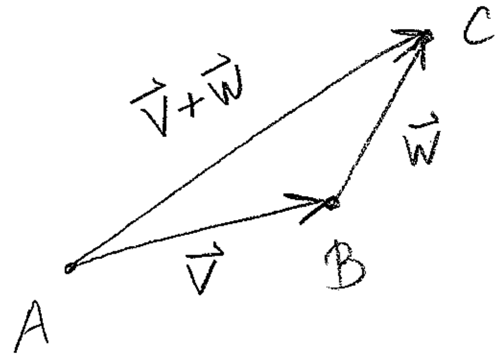


$\vec{v} = \vec{w}$

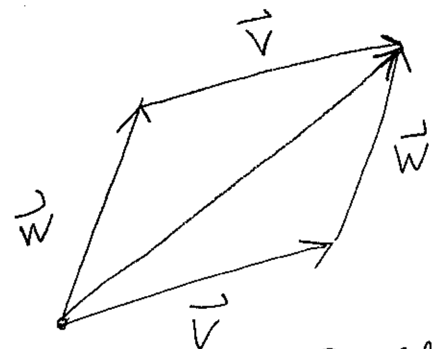
Addition:



Start at A, head along $\vec{v}$ to B, then along $\vec{w}$ to C.



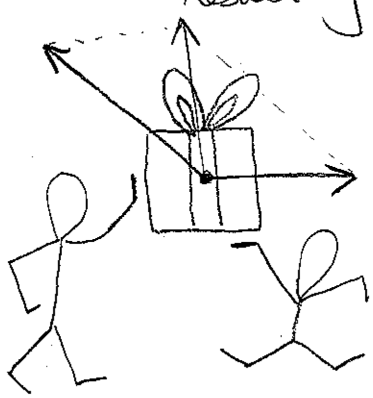
Note



and so

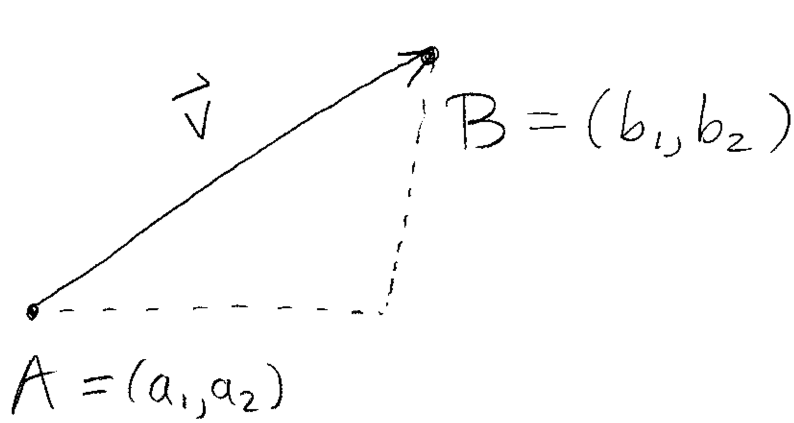
$$\boxed{\vec{v}+\vec{w} = \vec{w}+\vec{v}}$$

Force:



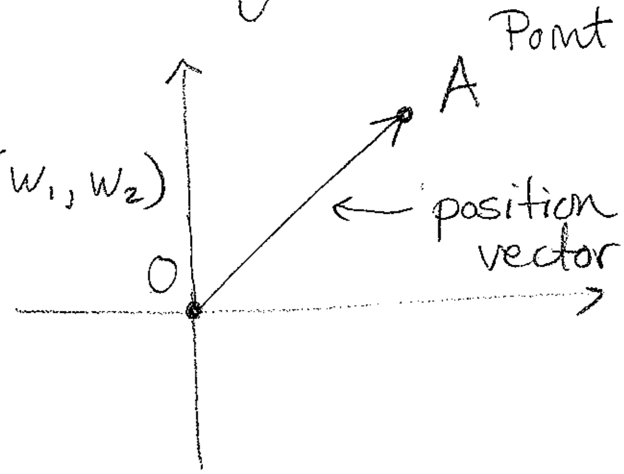
Resulting force is the sum of the two forces.

Like points in $\mathbb{R}^2$, vectors are param. by pairs of #s.



$$\vec{v} = (b_1 - a_1,\ b_2 - a_2)$$

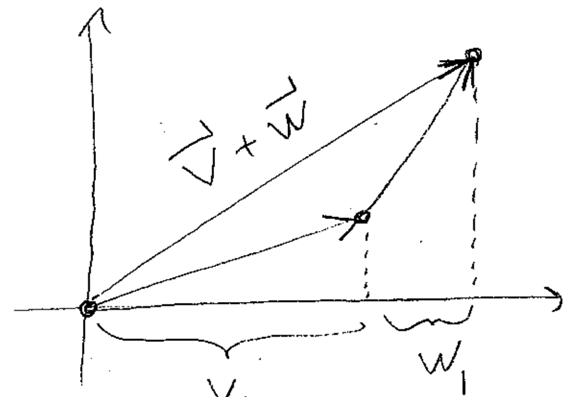
Points and vectors are diff. though.



If $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$

then

$$\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2)$$

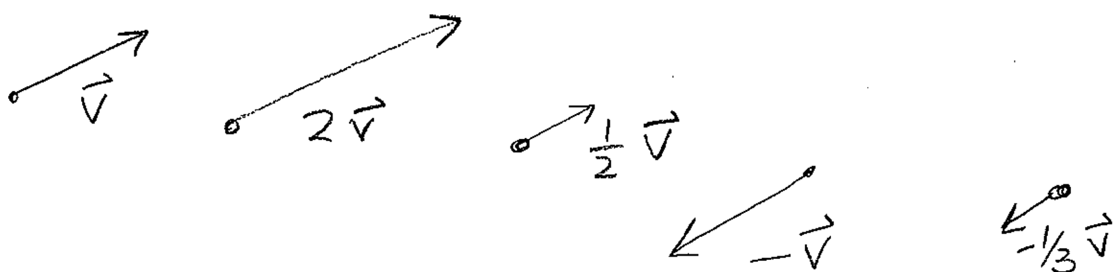


<u>Scalar multiplication:</u>

$c\,\vec{v}$ = same direction, length scaled by c.

c: #

$\vec{v}$: vector

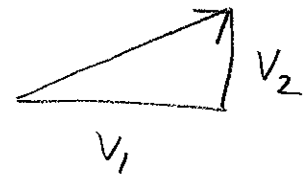


In coordinates

$$c\vec{v} = (cv_1, cv_2) \quad \text{where } \vec{v} = (v_1, v_2)$$

Length: $|\vec{v}| = \sqrt{v_1^2 + v_2^2}$

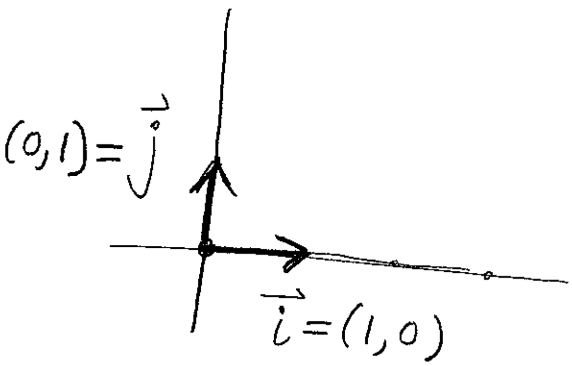
(Some books use $\|\vec{v}\|$.)

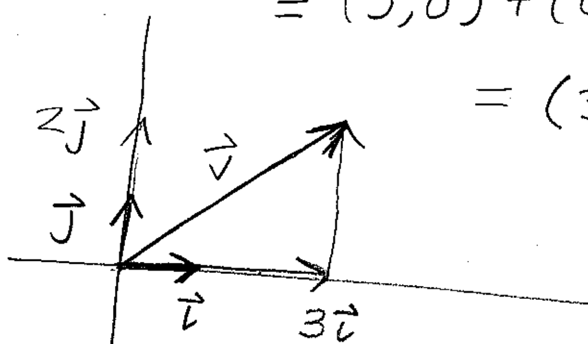


Standard Vectors: $\vec{v} = 3\,\vec{i} + 2\,\vec{j} = 3\cdot(1,0) + 2(0,1)$

$= (3,0) + (0,2)$

$= (3,2)$





Properties: [Can work out from geometry or algebra.]

$\vec{v} + \vec{w} = \vec{w} + \vec{v}$ $\qquad (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

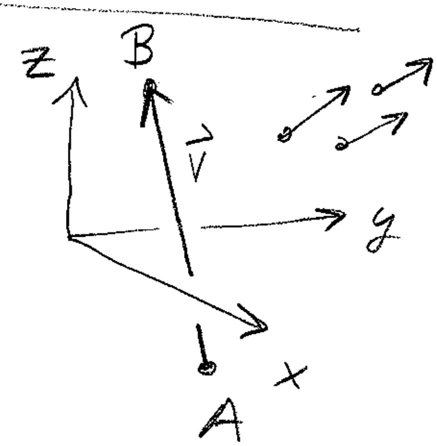
$\vec{v} + \vec{0} = \vec{v}$ if $\vec{0} = (0,0)$ $\qquad \vec{v} + (-1)\vec{v} = \vec{0}$

$c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$ $\qquad (c+d)\vec{v} = c\vec{v} + d\vec{v}$

$(cd)\vec{v} = c(d\vec{v})$ $\qquad 1\vec{v} = \vec{v}$

Vectors in $\mathbb{R}^3$: Same idea

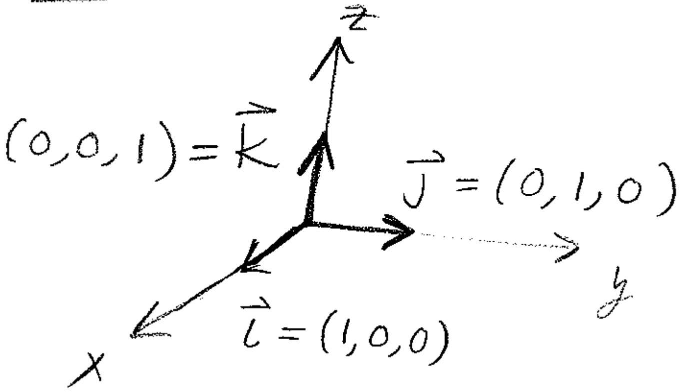
$\vec{v} = (v_1, v_2, v_3)$



Vectors in $\mathbb{R}^n$:

Once more, with feeling.

Standard vectors in $\mathbb{R}^3$:



$$(1,3,2) = \vec{i} + 3\vec{j} + 2\vec{k}$$

[Can we multiply vectors? In $\mathbb{R}^3$ there are two ways to do this, different from each other and also multiplication of numbers.]

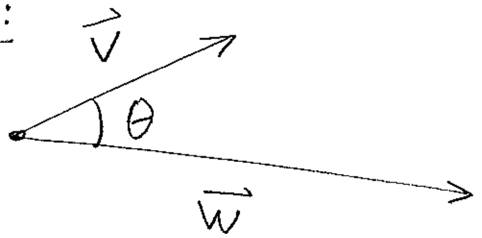
Dot Product: $\vec{v} = (v_1, v_2, v_3)$ $\vec{w} = (w_1, w_2, w_3)$ vectors in $\mathbb{R}^3$.

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Ex: $\vec{v} = (1, 2, 0)$, $\vec{w} = (-1, 0, +2)$

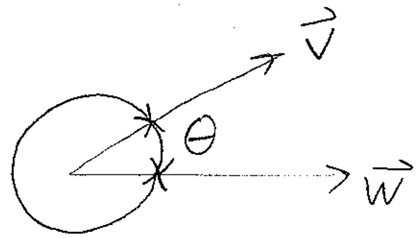
$$\vec{v} \cdot \vec{w} = 1 \cdot (-1) + 2 \cdot 0 + 0 \cdot 2 = -1$$

Key:

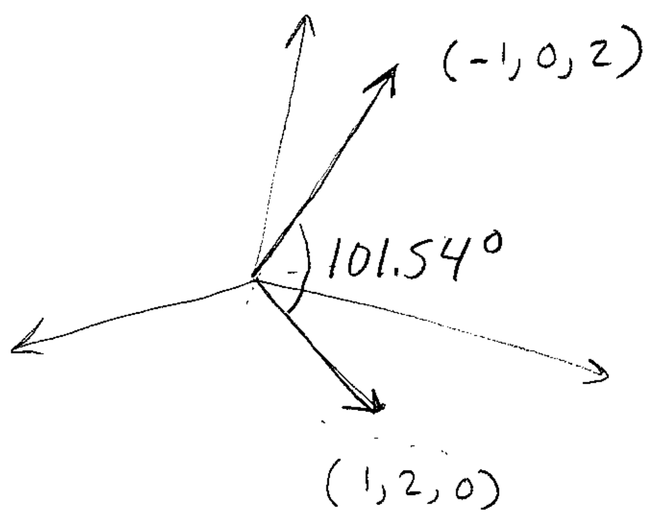


$$\boxed{\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}| \cos\theta}$$

$0 \leq \theta \leq \pi$ is the smaller of two angles.



Ex: $\cos\theta = \dfrac{\vec{v} \cdot \vec{w}}{|\vec{v}||\vec{w}|} = \dfrac{-1}{5} \Rightarrow \theta \approx 101.54°$



Note: Thus

$$\vec{v} \cdot \vec{w} = 0$$

exactly when $\cos\theta = 0$,

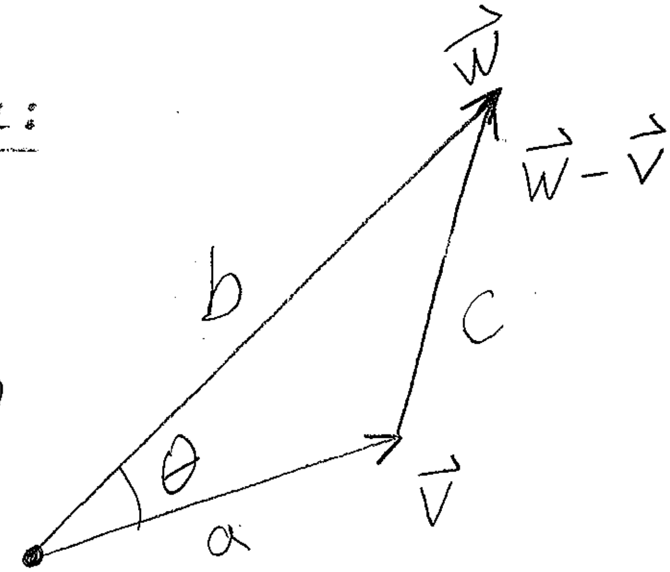
i.e. $\theta = \pm \pi/2$ and the vectors meet at right angles.

Properties [Easy to see from the def.]

$$|\vec{v}|^2 = \vec{v} \cdot \vec{v} \qquad \vec{u} \cdot (\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v} + \vec{u}\cdot\vec{w}$$

$$\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v} \qquad (c\,\vec{v})\cdot\vec{w} = c\,(\vec{v}\cdot\vec{w})$$

Idea behind key formula:



Law of cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

So:

$$c^2 = |\vec{w}-\vec{v}|^2 = (\vec{w}-\vec{v})\cdot(\vec{w}-\vec{v})$$
$$= \vec{w}\cdot\vec{w} - \vec{w}\cdot\vec{v} - \vec{v}\cdot\vec{w} + \vec{v}\cdot\vec{v}$$
$$= |\vec{w}|^2 + |\vec{v}|^2 - 2\,\vec{v}\cdot\vec{w}$$
$$= a^2 + b^2 - 2\,\vec{v}\cdot\vec{w}$$

Thus $\vec{v}\cdot\vec{w} = ab\cos\theta$

$$= |\vec{v}||\vec{w}|\cos\theta.$$