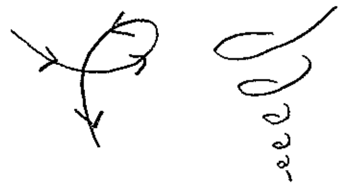
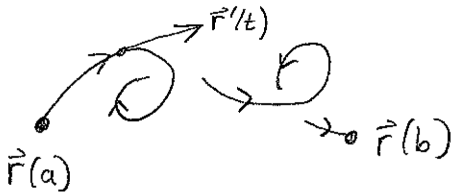
Lecture 18: Integration along curves (13.3 and 16.2)

Last time: $\vec{r}: \mathbb{R} \to \mathbb{R}^2$ or $\mathbb{R}^3$



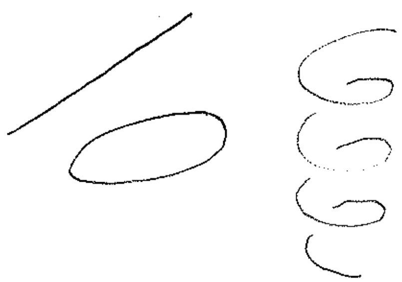
Length: $\vec{r}: [a,b] \to \mathbb{R}^3$



$$\text{Length} = \int_a^b |\vec{r}'(t)|\,dt$$

Really two concepts:

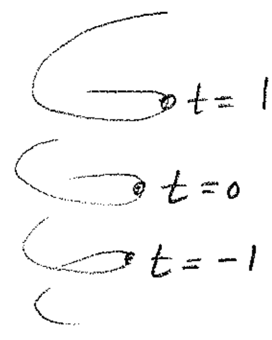
Curve: A set of pts in $\mathbb{R}^3$ looking like



Parameterization:

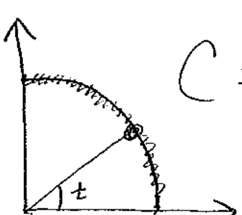
$\vec{r}: \mathbb{R} \to \mathbb{R}^3$

Instructions for moving along a curve



[Any curve has many parameterizations.]

Ex: $C = 1/4$ of a unit circle in $\mathbb{R}^2$



(a) $\vec{r}: [0, \pi/2] \to \mathbb{R}^2$

$$\vec{r}(t) = (\cos t, \sin t)$$

$$\vec{r}'(t) = (-\sin t, \cos t)$$

$$|\vec{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t} = 1$$

$$\text{Length} = \int_0^{\pi/2} |\vec{r}'(t)|\,dt = \int_0^{\pi/2} 1\,dt = \boxed{\pi/2}$$

(b) $\vec{r}:[0,1] \to \mathbb{R}$ $\quad \vec{r}(t) = (t, \sqrt{1-t^2})$

$$\vec{r}\,'(t) = \left(1, \frac{-t}{\sqrt{1-t^2}}\right)$$

$$|\vec{r}\,'(t)| = \sqrt{1 + \frac{t^2}{1-t^2}} = \frac{1}{\sqrt{1-t^2}}$$

Length

$$= \int_0^1 \frac{1}{\sqrt{1-t^2}} = \arcsin(t)\Big|_{t=0}^{1} = \pi/2 - 0 = \boxed{\pi/2} \checkmark$$

<u>Integration along a curve:</u> (Sect 16.2)

C curve in $\mathbb{R}^2$

$f: \mathbb{R}^2 \to \mathbb{R}$ a fn

$$\int_C f\,ds = \int_a^b f(r(t))\,|\vec{r}\,'(t)|\,dt$$

where $\vec{r}:[a,b] \to \mathbb{R}^2$ is a param. of C.

[turns out <u>not</u> to depend on $\vec{r}$, just C.]

<u>Some meanings</u>

(1) f = temperature

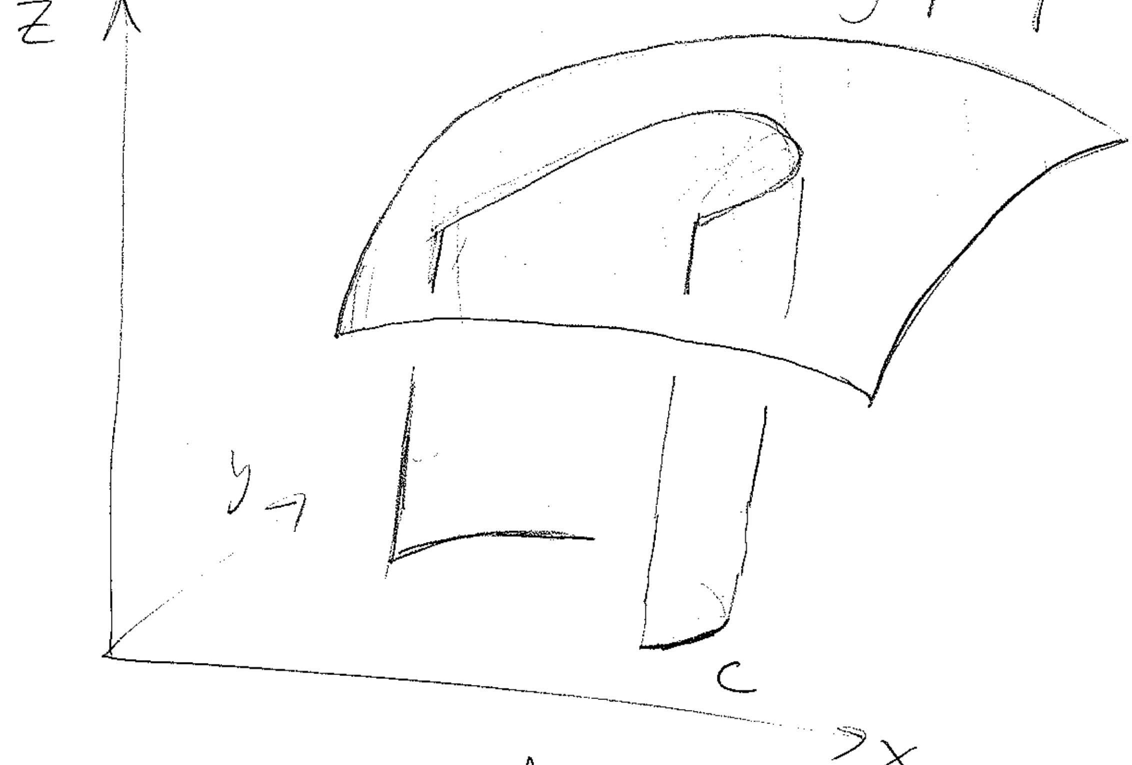
<u>Compare</u>: Average of $f:\mathbb{R}\to\mathbb{R}$ on $[a,b]$ is $\frac{1}{b-a}\int_a^b f(x)\,dx$

$$\text{Average temp along } C = \frac{1}{\text{Length } C}\int_C f\,ds$$

② f = density of material curve is made of $\left({}^{\text{mass}}/_{\text{length}} \right)$

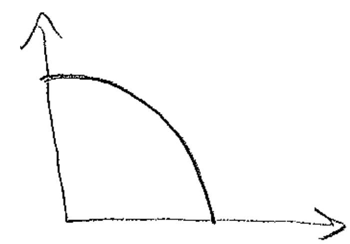
$$\text{Mass of curve} = \int_C f \, ds$$

③ Area of region above C and below the graph of f



<u>Ex</u>: Find average of $f(x,y) = x$ on

$\vec{r}(t) = (\cos t, \sin t) \quad 0 \leq t \leq \pi/2$

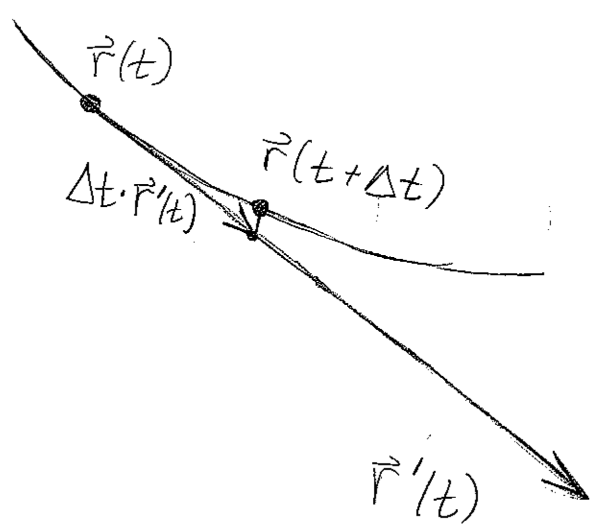


$$\int_C f \, ds = \int_0^{\pi/2} f(\vec{r}(t)) \, |\vec{r}\,'(t)| \, dt$$

$$= \int_0^{\pi/2} \cos(t) \cdot 1 = \sin(t) \Big|_{t=0}^{\pi/2} = 1 - 0 = 1$$

$$\text{Average} = \frac{\int_C f\,ds}{\text{Length}} = \frac{2}{\pi} \approx 0.6366$$

<u>Linear Approximation</u>: $\vec{r}: \mathbb{R} \to \mathbb{R}^3$



$$\vec{r}(t+\Delta t) = \vec{r}(t) + \Delta t\, \vec{r}'(t) + E(t)$$

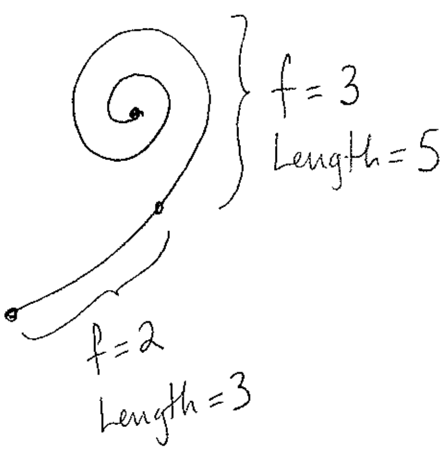
Provided $\vec{r}'(t)$ exists, will have

$$\lim_{t \to 0} \frac{E(t)}{t} = \vec{0}$$

Alternatively, $\vec{r}(t+\Delta t) - \vec{r}(t) \approx \Delta t\, \vec{r}'(t)$

Understanding these integrals.

<u>Ex.</u>

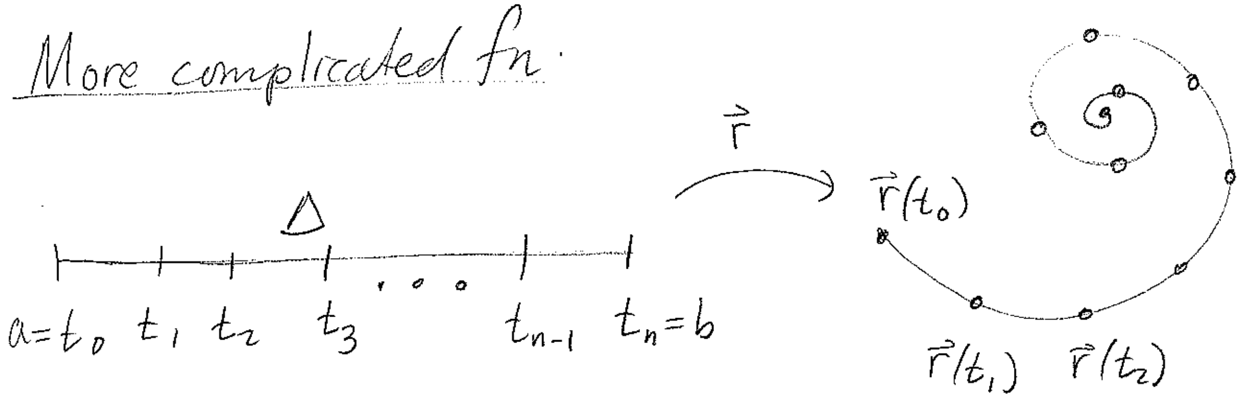


$$\begin{array}{c}\text{Average of} \\ f \text{ on } C\end{array} = \begin{pmatrix}\text{portion} \\ \text{where} \\ f=2\end{pmatrix} \cdot 2 + \begin{pmatrix}\text{portion} \\ \text{where} \\ f=3\end{pmatrix} \cdot 3$$

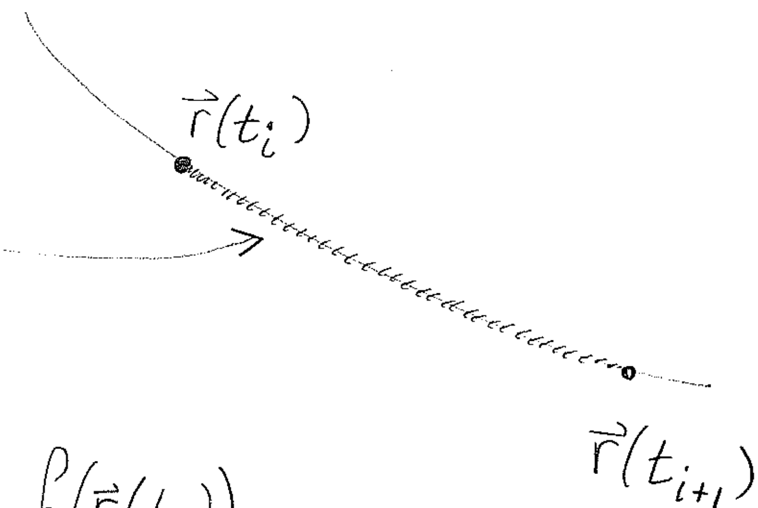
$$= \left(\frac{3}{8}\right) \cdot 2 + \left(\frac{5}{8}\right) \cdot 3$$

$$= \frac{21}{8} = 2\tfrac{5}{8}$$

More complicated fn:



If Δt is small and f is continuous, then f is almost constant on



So this segment contributes

$$\approx \frac{(\text{length of segment})}{(\text{length of } C)} \cdot f(\vec{r}(t_i))$$

to the average. Now the seg. has length $\approx |\vec{r}\,'(t_i)|\Delta t$ and so

$$\text{Average} \approx \sum_{i=0}^{n-1} \frac{(\text{len of } i^{th} \text{ seg})}{(\text{len of } C)} f(\vec{r}(t_i))$$

$$\approx \frac{1}{\text{len}(C)} \sum_{i=0}^{n-1} f(\vec{r}(t_i))\,|\vec{r}\,'(t_i)|\,\Delta t$$

As $\Delta t \to 0$, we get

$$\text{Average} = \frac{1}{\text{Len}(C)} \int_0^b f(r(t))\, |\vec{r}\,'(t)|\, dt$$

$$= \frac{1}{\text{Len}(C)} \int_C f\, ds.$$

<u>Compare</u>: Average of f on $[a,b]$ $= \dfrac{1}{b-a} \displaystyle\int_a^b f(t)\, dt$