Lecture 17: Introduction to space curves (13.1 and 13.2) + 13.4

Next two weeks: "Curves in space..."

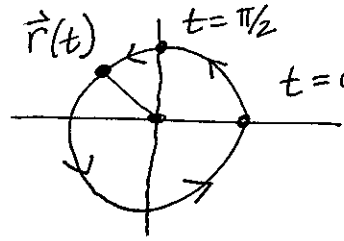
Chapter 13, Section 16.1 – 16.3

Last time: Not really relevant.

Curve: $\vec{r} : \mathbb{R} \to \mathbb{R}^2$ or $\mathbb{R}^3$ (vector-valued function)

Ex: $\vec{r} : \mathbb{R} \to \mathbb{R}^2$

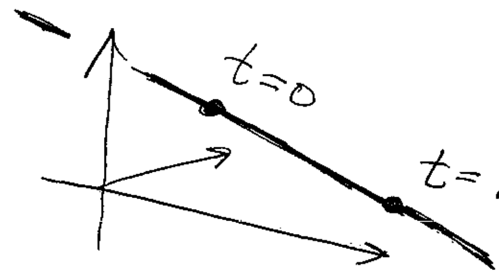
$$\vec{r}(t) = (\cos t, \sin t)$$



Circle

Ex: $\vec{r} : \mathbb{R} \to \mathbb{R}^3$

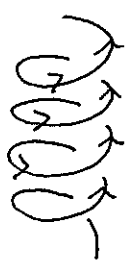
$$\vec{r}(t) = (1,0,1) + t(2,1,-1)$$
$$= (1+2t, t, 1-t)$$



Line

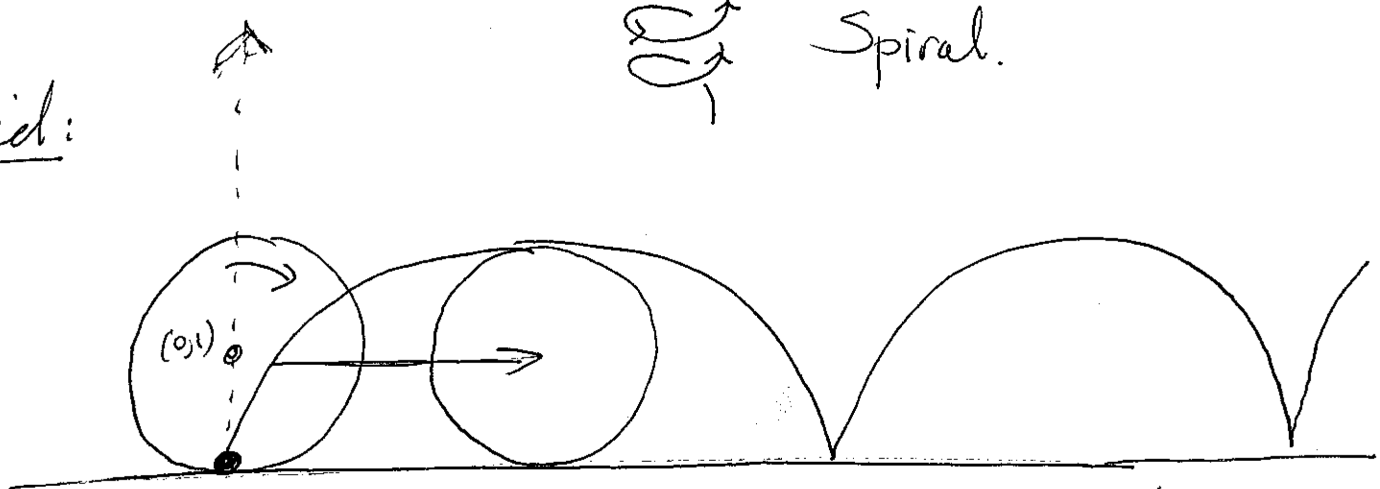
Ex: $\vec{r} : \mathbb{R} \to \mathbb{R}^3$

$$\vec{r}(t) = (\cos t, \sin t, t)$$



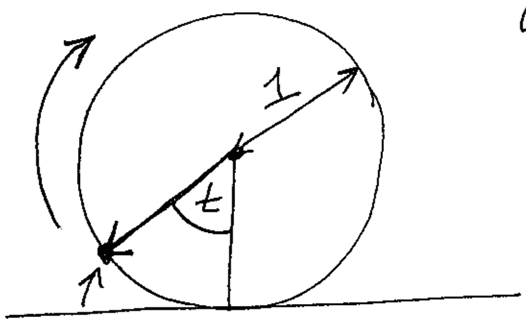
Helical Spiral.

Cycloid:



Suppose a wheel with radius 1 rotates at 1 radian/sec

Thus after time t, the center has moved a distance t. So if center is at $(0,1)$ at $t=0$, its at $(t,1)$. Combined with the rotation,



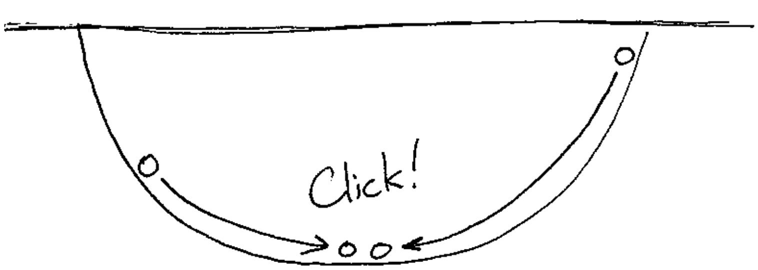
vector is $(-\sin t, -\cos t)$

we get

$$\vec{r}(t) = (t,1) + (-\sin t, -\cos t)$$
$$= (t - \sin t, -\cos t)$$

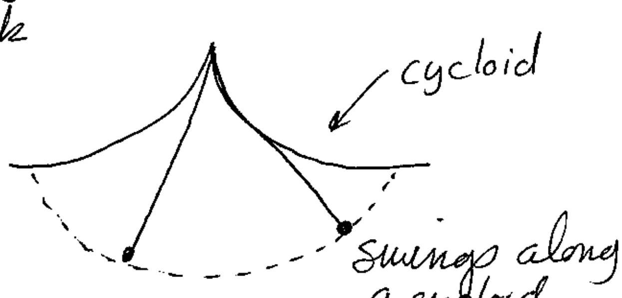
Upside down, this curve has the following remarkable properties:

Tautochrone

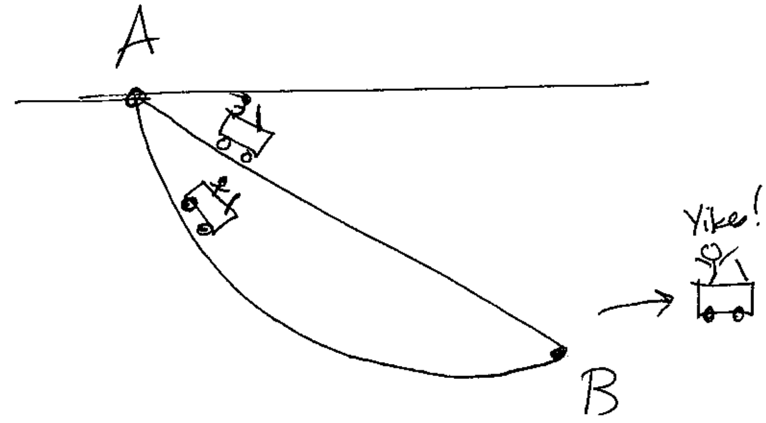


Same time to bottom

Huygens (1650s) pendulum clock



Brachistochrone



minimizes time from A to B.

This is an <u>infinite dimensional</u> min/max problem.

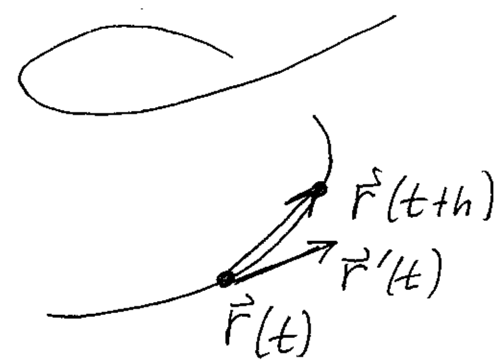
("Calculus of Variations")

Derivatives: $\vec{r}: \mathbb{R} \to \mathbb{R}^3$ $\quad \vec{r}(t) = (r_1(t), r_2(t), r_3(t))$

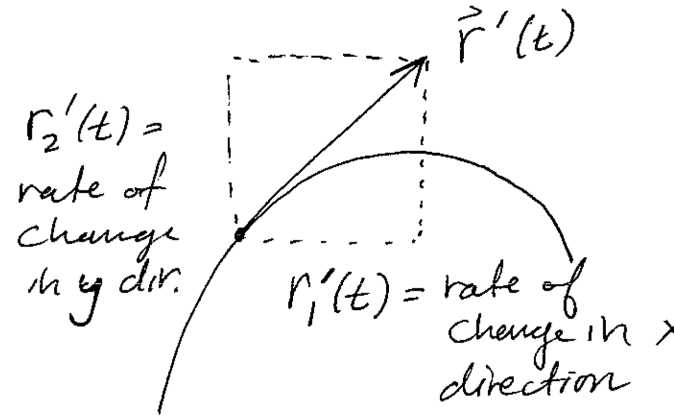
$\uparrow$ component fns.

$$\vec{r}\,'(t) = (r_1'(t), r_2'(t), r_3'(t))$$

$$= \lim_{h \to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

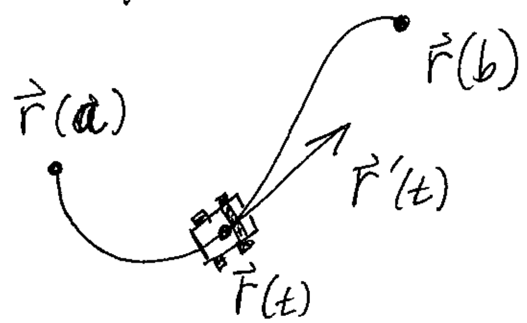


The vector $\vec{r}\,'(t)$ is tangent to the curve at $\vec{r}(t)$. Geometrically, it is the velocity vector of the motion at time t. (notice the units of $r'(t)$ are $\frac{\text{distance}}{\text{time}}$). The Speed of the motion at time t is $|\vec{r}(t)|$



Also $\vec{r}\,': \mathbb{R} \to \mathbb{R}^3$ and $\vec{r}\,''(t)$ is the acceleration at time t.

Length of a curve: $\vec{r} : [a, b] \longrightarrow \mathbb{R}^2 \text{ (or } \mathbb{R}^3\text{)}$

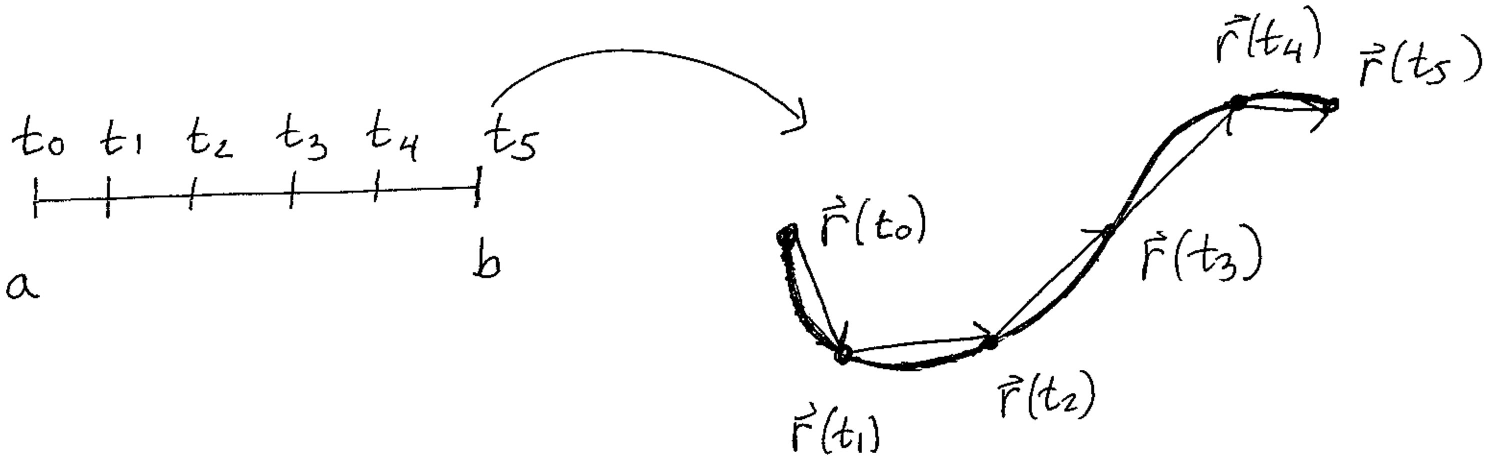


How long is this path?

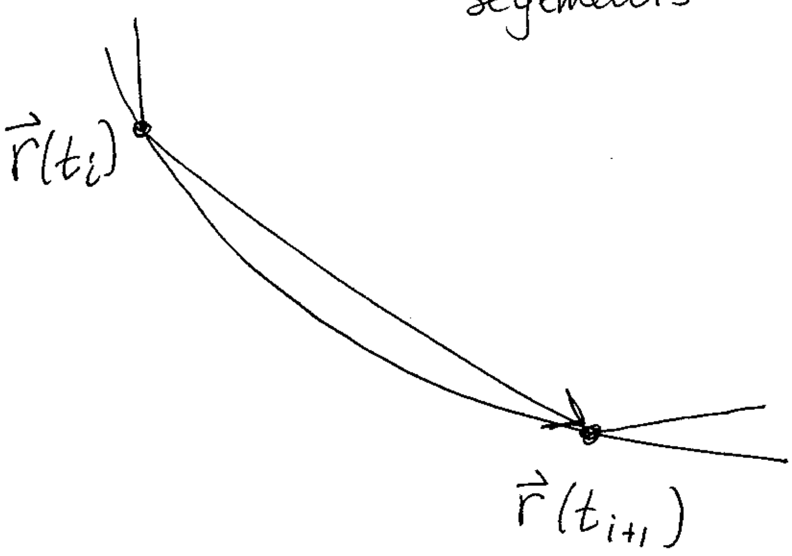
If we view this the motion of something, have:

$$\begin{array}{c}\text{distance}\\ \text{traveled}\end{array} = \int_a^b \begin{array}{c}\text{speed}\\ \text{at time } t\end{array} \, dt = \int_a^b |\vec{r}'(t)| \, dt$$

Annother point of view: Approximate by straight segments



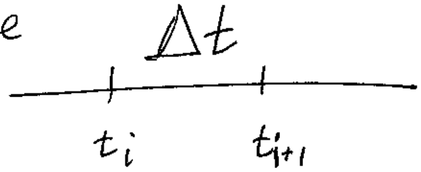
$$\text{Length} \approx \sum \begin{array}{c}\text{length of}\\ \text{segements}\end{array} = \sum_{i=0}^{n} |\vec{r}(t_{i+1}) - \vec{r}(t_i)|$$



Linear approximation: (just like in one var.)

$$\vec{r}(t_i + \Delta t) \approx \vec{r}(t_i) + \vec{r}\,'(t_i)\,\Delta t$$

Take



and so

$$\left|\vec{r}(t_{i+1}) - \vec{r}(t_i)\right| \approx \left|\vec{r}\,'(t_i)\right| \Delta t.$$

So

$$\text{Length} \approx \sum_{i=0}^{n} \left|\vec{r}(t_{i+1}) - \vec{r}(t_i)\right| \approx \underbrace{\sum_{i=0}^{n} \left|\vec{r}\,'(t_i)\right| \Delta t}_{\text{Riemann Sum}}$$

As $\Delta t \to 0$, these Riemann sums converge to

$\int_a^b |\vec{r}\,'(t)|\,dt$, and also to the length.

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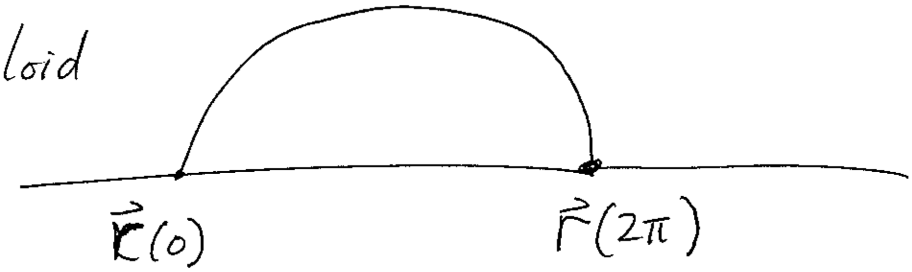
Ex: Cycloid



$$\vec{r}(t) = (t - \sin t, 1 - \cos t)$$

$$\vec{r}\,'(t) = (1 - \cos t, \sin t)$$

$$|\vec{r}\,'(t)| = \sqrt{\underbrace{(1-\cos t)^2}_{1 - 2\cos t + \cos^2 t} + \sin^2 t} \quad \cancel{= \sqrt{2-2}} $$

$$= \sqrt{2 - 2\cos t}$$

$$\text{Length} = \int_0^{2\pi} |\vec{r}\,'(t)|\,dt = \int_0^{2\pi} \sqrt{2}\cdot\sqrt{1-\cos t}\,dt$$

$$= \int_0^{2\pi} \sqrt{2}\,\sqrt{2\sin^2 \tfrac{t}{2}}\,dt$$

$$\boxed{\begin{aligned} &\cos t = 1 - 2\sin^2 \tfrac{t}{2} \\ &\cos\left(\tfrac{t}{2}+\tfrac{t}{2}\right) = \cos\tfrac{t}{2}\cdot\cos\tfrac{t}{2} - \sin\tfrac{t}{2}\cdot\sin\tfrac{t}{2} \\ &\qquad = \cos^2\tfrac{t}{2} - \sin^2\tfrac{t}{2} \end{aligned}}$$

$$= \int_0^{2\pi} 2\sin\frac{t}{2}\,dt \quad \left(\text{since } \sin\tfrac{t}{2} \text{ is } > 0 \text{ on } [0, 2\pi]\right)$$

$$= -4\cos\frac{t}{2}\Big|_0^{2\pi} = 4 - (-4) = 8.$$