

Lecture 16: Constrained min/max (§14.8)

Last time:

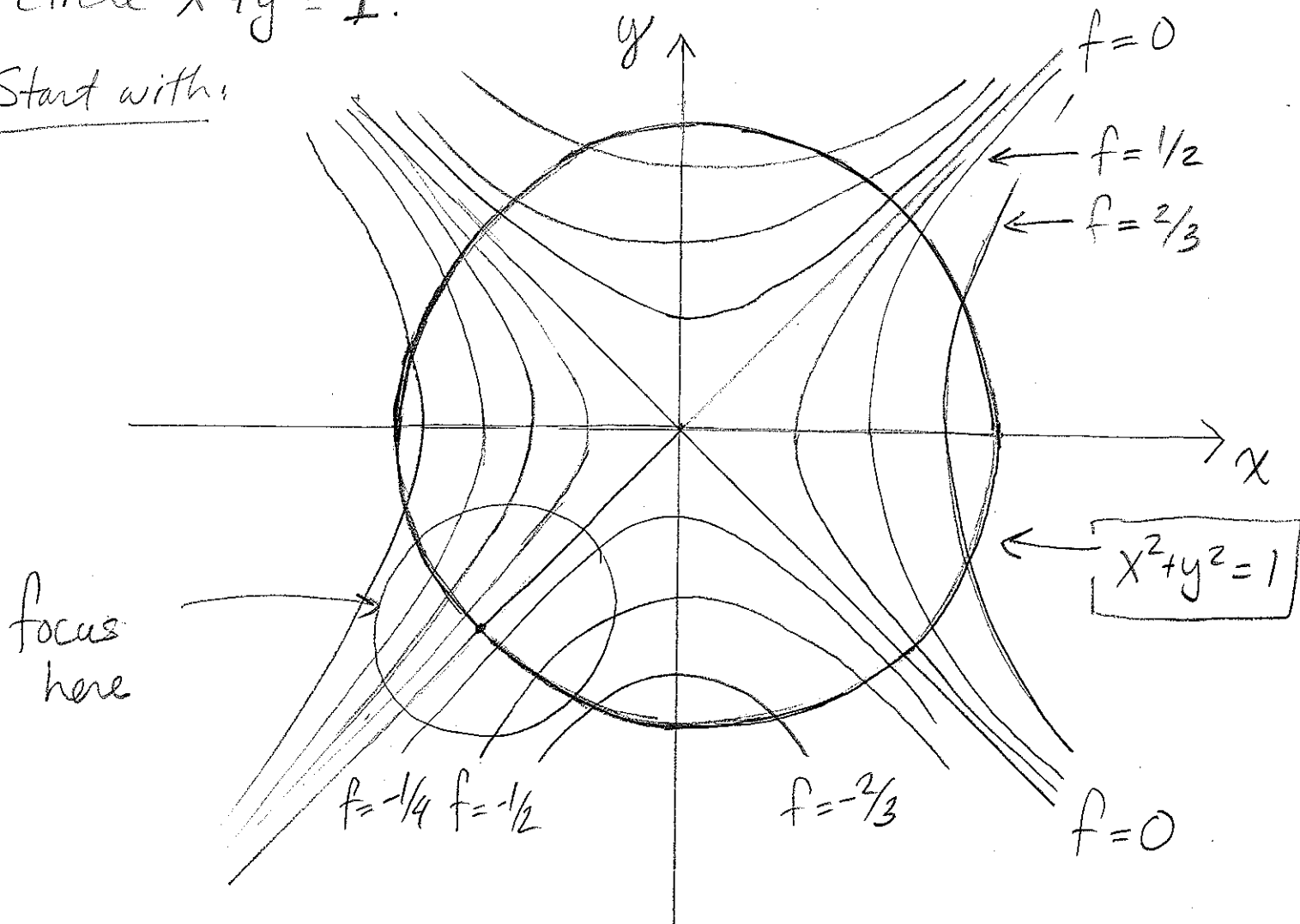
Extreme Value Thm: f continuous on D in \mathbb{R}^n .

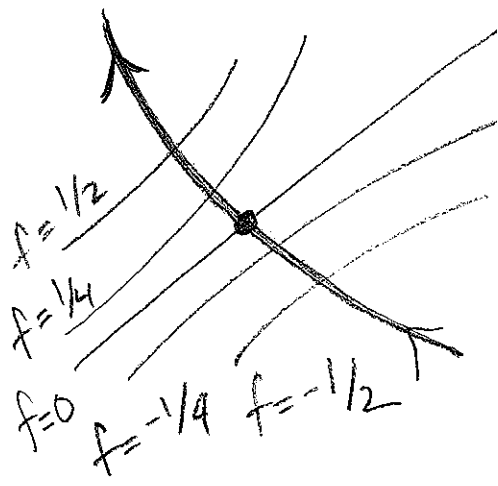
If D is closed and bounded, then f has both an absolute min and an absolute max on D .

These occur at critical pts of f or on the boundary of D .
↑ focus today.

Ex: Find the max of $f(x,y) = x^2 - y^2$ on the unit circle $x^2 + y^2 = 1$.

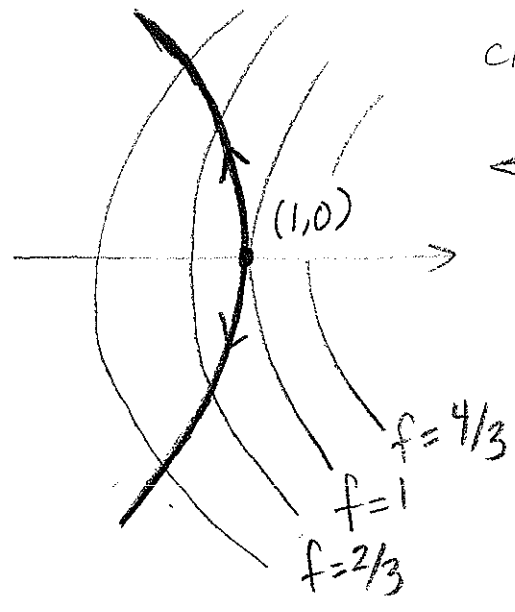
Start with:





When we have this picture, don't have a local max since can increase f by moving clockwise along the circle.

However, when the level set of f is tangent to the circle, we can have a local max.



← Here, f decreases as we move away from $(1,0)$ in either direction.

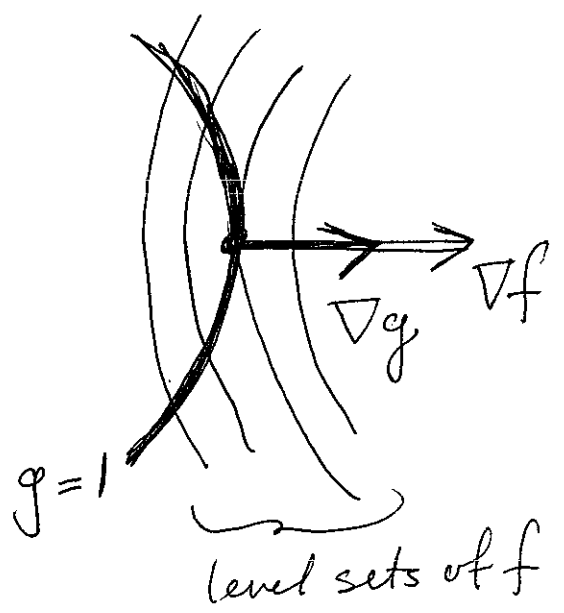
Four tangencies in this example:

	$(1,0)$	$(-1,0)$	$(0,1)$	$(0,-1)$
Value of f	1	1	-1	-1
Type	max	max	min	min

absolute in each case; the circle is closed and bounded.

Finding these tangencies in general:

View the circle as the level set $g(x,y) = 1$ where $g(x,y) = x^2 + y^2$. At a tangency, ∇f is at right angles to the circle:



Key: At a tangency, ∇f and ∇g point in the same direction:

$$\nabla f = \lambda \nabla g$$

↑
some number.

Ex: $g(x,y) = x^2 + y^2 = 1$ Lagrange Multipliers
discovered by Euler

$$\nabla f = (2x, -2y) = \lambda \nabla g = \lambda (2x, 2y) = (2\lambda x, 2\lambda y)$$

So $2x = 2\lambda x$ and $-2y = 2\lambda y$.

If $x \neq 0$, then $\lambda = 1$ and $y = 0 \Rightarrow x = \pm 1$.

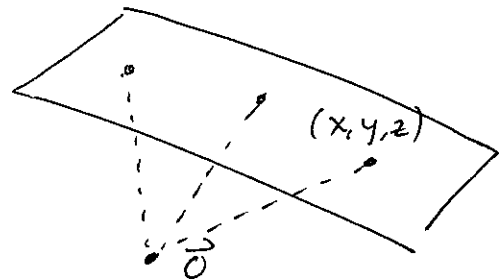
If $y \neq 0$, then $\lambda = -1$ and $x = 0 \Rightarrow y = \pm 1$
 ↑
 from $x^2 + y^2 = 1$.

So the critical points are $(1,0)$ $(-1,0)$ $(0,1)$ $(0,-1)$ 506
just as we found before.

Ex: Find the distance from $\overbrace{x-y+2z=3}^{g(x,y,z)}$ to $\vec{0}$

Minimize: $f(x,y,z) = x^2 + y^2 + z^2$

Subject to: $g(x,y,z) = 3$



Crit pts: $\nabla f = \lambda \nabla g = (\lambda, -\lambda, 2\lambda)$

$\nabla f = (2x, 2y, 2z)$

$\nabla g = (1, -1, 2)$

$2x = \lambda$

$2y = -\lambda \Rightarrow y = -x$

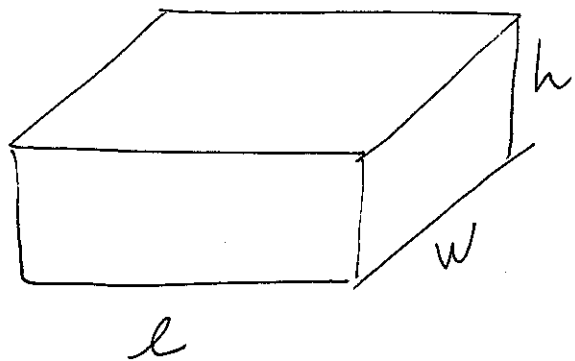
$2z = 2\lambda \Rightarrow z = 2x$

Combine with $g = 3$ gives $x - (-x) + 2(2x) = 3$
and so $x = 1/2$, and $y = -1/2$, $z = 1$ just as
on Monday.

Key Features: ① Algebra easier than before.

② Don't have to solve for one var, which we
won't be able to do for complicated g

Ex: Find the rectangular box of area 6 and largest volume. [What do you expect the ans to be?] 51



Maximize: $V = lwh$

Subject to: $A = 2lh + 2lw + 2wh = 6.$

$$\nabla V = (wh, lh, lw) = \lambda \nabla A = \lambda 2(h+w, l+h, l+w)$$

$$\Rightarrow \frac{1}{2\lambda} = \frac{1}{w} + \frac{1}{h} = \frac{1}{h} + \frac{1}{l} = \frac{1}{w} + \frac{1}{l}$$

$$\Rightarrow \frac{1}{l} = \frac{1}{w} = \frac{1}{h} \Rightarrow l = w = h$$

Q: Why does this crit pt have to be a max?

Combine with $A = 6$ gives $6l^2 = 6 \Rightarrow l = w = h = 1.$

Point: V and A is very symmetric. In particular

If (l_0, w_0, h_0) is a crit pt, so is (w_0, h_0, l_0) and $(w_0, l_0, h_0), \dots$ Hence if there is only one crit pt, it must be symmetric.