# Lecture 15: More on min/max (14.7)

**Last time:** $f : \mathbb{R}^2 \to \mathbb{R}$ with $(a,b)$ a crit pt.

$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

If $D > 0$ and

$f_{xx}(a,b) > 0 \Rightarrow$ local min

$f_{xx}(a,b) < 0 \Rightarrow$ local max

If $D < 0 \Rightarrow$ saddle

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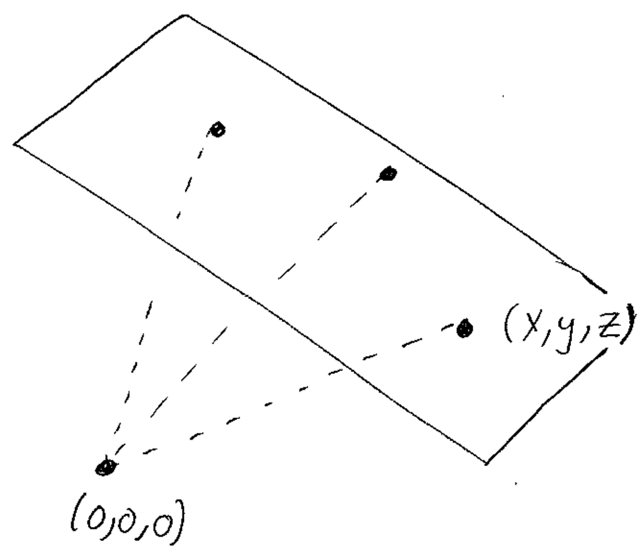
**Ex:** Find the distance from $(0,0,0)$ to the plane $x - y + 2z = 6$.

$$z = \frac{6-x+y}{2}$$




**Need to minimize:**

$$f(x,y) = \begin{pmatrix} \text{distance from} \\ (x, y, \frac{6-x+y}{2}) \\ \text{to } (0,0,0) \end{pmatrix}^2 = x^2 + y^2 + \frac{1}{4}(6-x+y)^2$$

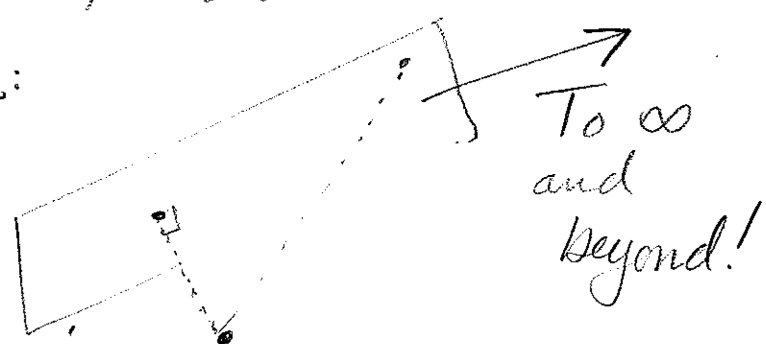
**Critical Points:** $\nabla f = \vec{0}$

$$\frac{\partial f}{\partial x} = 2x + \frac{1}{2}(6-x+y)\cdot(-1) = \frac{5}{2}x - \frac{1}{2}y - 3$$

$$\frac{\partial f}{\partial y} = 2y + \frac{1}{2}(6-x+y) = -\frac{1}{2}x + \frac{5}{2}y + 3$$

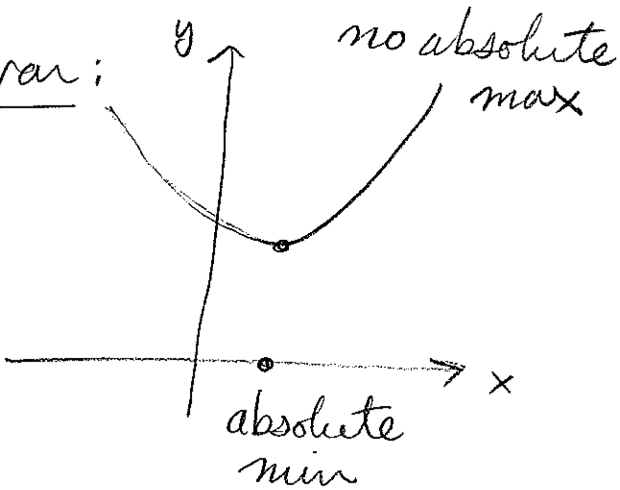
Setting $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ we find a unique solution $x = 1$ and $y = -1$. Since this is the only crit pt, it must be the minimum and so the distance is $\sqrt{f(1,-1)} = \sqrt{1^2+1^2+\frac{1}{4}(4)^2} = \sqrt{6}$.

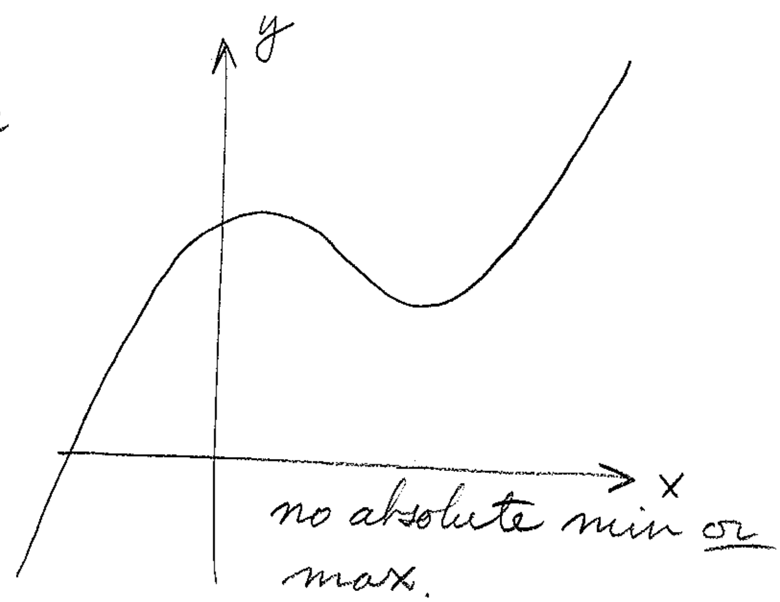
Hey, the same reasoning says that $f$ takes its maximum at $(1,-1)$!? In fact, $f$ has no max, as clear geometrically:




In this case it seems like there should be a min, but let's look at gen. criteria for absolute min/max to exist.

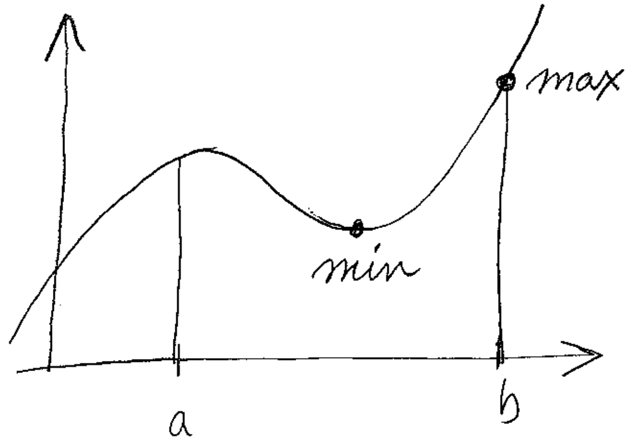
One var:



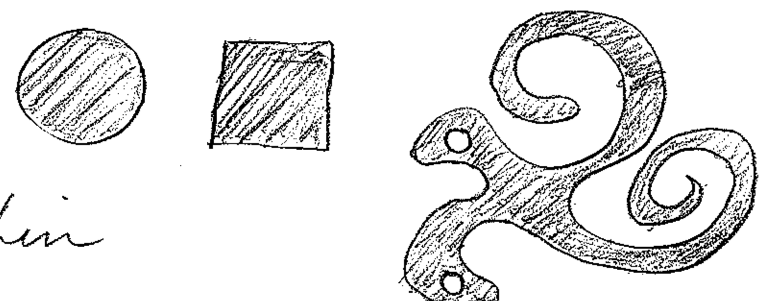



**Extreme value theorem:** $f$ continuous on $[a,b] = \{a \le x \le b\}$. Then $f$ has both an absolute min and max

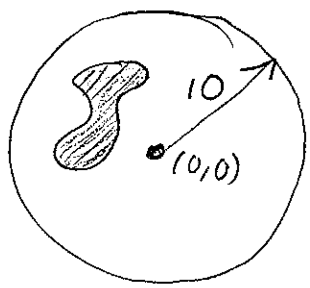
These min/max occure at either ⓐ a crit pt of $f$.
ⓑ one of $a$ or $b$.




**Two Var:** $D$ subset of $\mathbb{R}^2$:



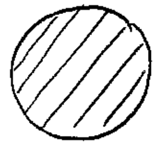
**Bounded:** Contained within a disk:



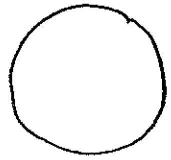

**Closed:** Contains all its boundary pts.

**Closed:**

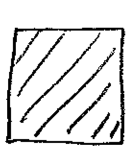
$x^2+y^2 \le 1$



$x^2+y^2 = 1$

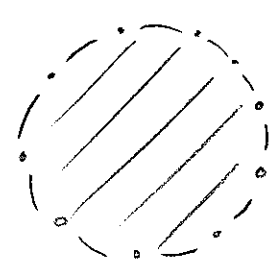


$\begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1 \end{cases}$

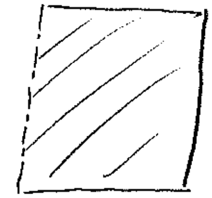


**Not Closed:**

$x^2+y^2 < 1$

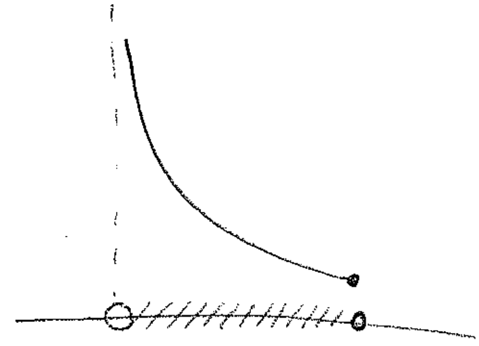


$\begin{cases} 0 < x \le 1 \\ 0 \le y \le 1 \end{cases}$



Closed important in 1 var too

$f = \frac{1}{x}$ on $(0,1] = \{0 < x \leq 1\}$



<u>Extreme Value Theorem:</u> $f$ continuous on $D$ in $\mathbb{R}^n$.

If $D$ is closed and bounded, then $f$ has absolute min and max on $D$. These occur either at crit pts of $f$ or on the boundary of $D$.

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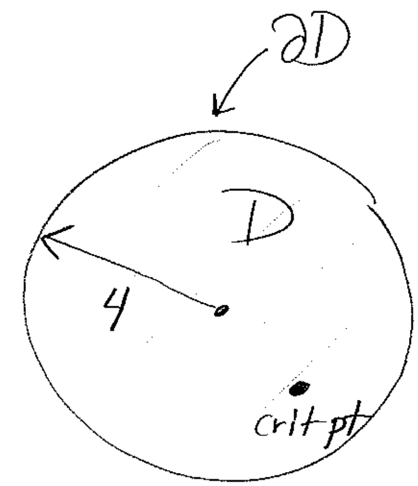
Back to orig. prob.

$$f(x,y) = x^2 + y^2 + \frac{1}{4}(6 - x + y)^2$$

which has one crit pt at $(1,-1)$.




Consider $D = \{x^2 + y^2 \leq 16\}$.

On $D$, $f$ has one crit pt and there $f = 6$.

On $\partial D$, $f(x,y) \geq x^2 + y^2 = 16$. So 6 is the absolute min value of $f$ on $D$.

As $f(x,y) \geq 16$ outside of $D$, $6$ is the absolute min of $f$ on <u>all</u> of $\mathbb{R}^2$.

<u>Double Check:</u> $f_{xx} = 5/2$, $f_{xy} = -\frac{1}{2}$, $f_{yy} = +5/2$

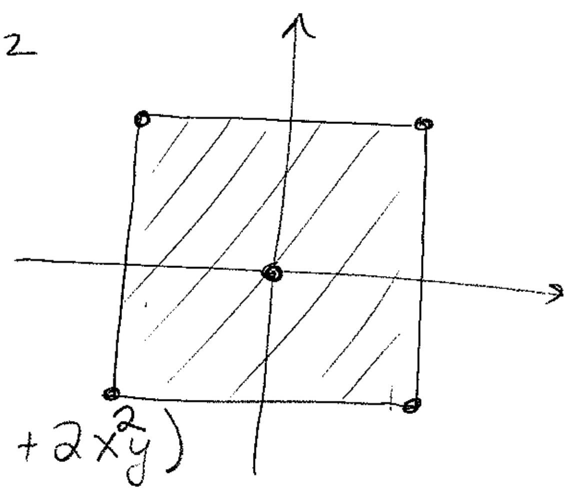
$$D = \begin{vmatrix} 5/2 & -1/2 \\ -1/2 & 5/2 \end{vmatrix} = \frac{25}{4} - \frac{1}{4} = 6 > 0 \text{ and } f_{xx} > 0.$$

[So our conclusion is consistent with the 2nd der. test.]

<u>Ex:</u> Find the absolute min/max of

$$f(x,y) = 1 - x^2 - y^2 + x^2y^2$$

on $D = \left\{ \begin{matrix} -1 \leq x \leq 1 \\ -1 \leq y \leq 1 \end{matrix} \right\}$



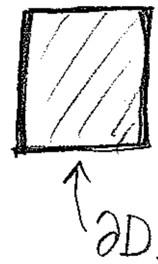
<u>Crit pts:</u> $\nabla f = (-2x + 2xy^2, -2y + 2x^2y)$

So $\nabla f = \vec{0} \iff x(y^2-1) = 0$ and $y(x^2-1) = 0$

Solutions are $(x,y) = (0,0), (1,1), (-1,1), (1,-1), (-1,-1)$

At $(0,0)$, $f = \underline{1}$.

On $\partial D$ we have $f = 0$ since

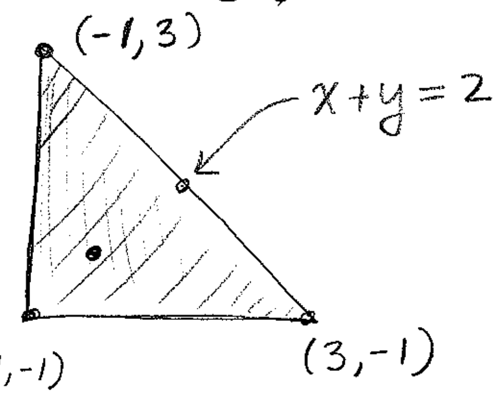



$$f(x, \pm 1) = 1 - x^2 - 1 + x^2(1) = 0$$

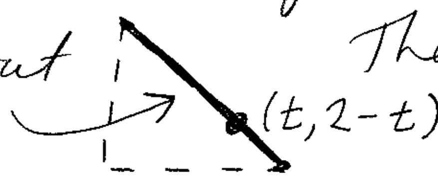
$$f(\pm 1, y) = 1 - 1 - y^2 + (1)\cdot y^2 = 0.$$

So absolute max of $f$ on $D$ is $1$ which occurs at $(0,0)$. Absolute min is $0$.

Ex: Same $f$, but $D =$



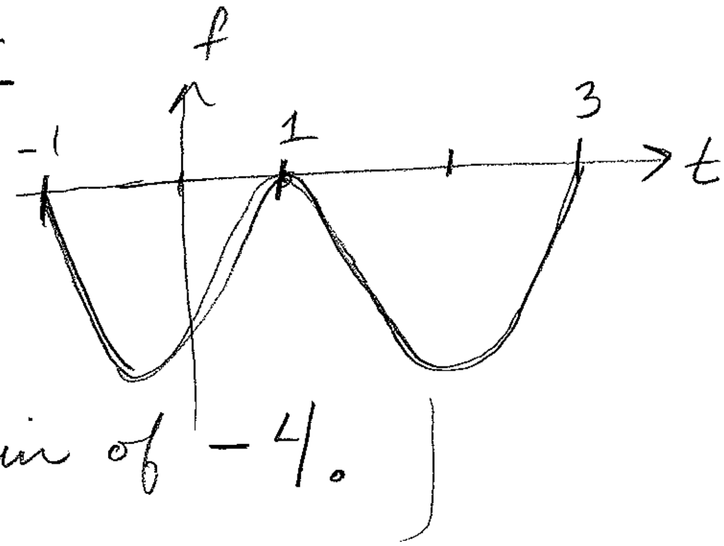

Still one crit pt inside $D$ and $f = 0$ on left and bottom sides of $\partial D$. What about




There have

$$f(t) = -3 + 4t + 2t^2 - 4t^3 + t^4$$

Which has min at $t = 1 \pm \sqrt{2}$ where $f = -4$.




So: On $D$ $f$ has abs max of $1$ and abs min of $-4$.