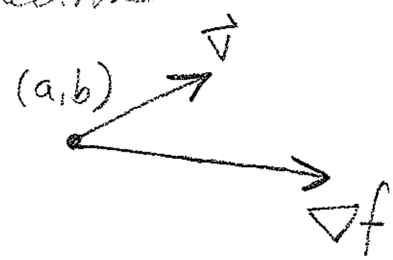
Lecture 12: Gradient (14.6) and Min/Max (14.7)

Last time: $f: \mathbb{R}^2 \to \mathbb{R}$

$D_{\vec{v}} f(a,b) =$ Rate f changes as we go in direction $\vec{v}$ starting from (a,b)



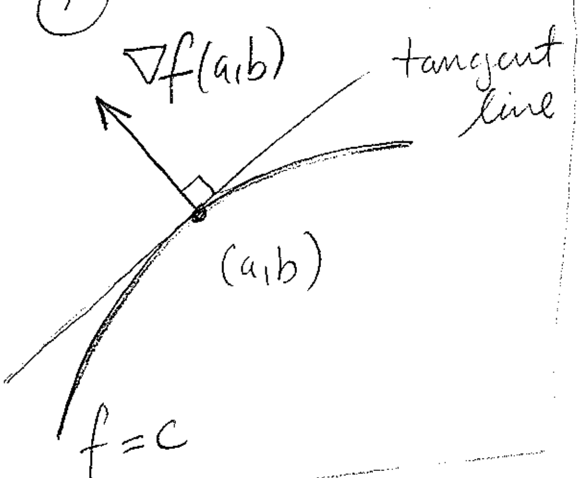
$$\nabla f(a,b) = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right)$$

Relationship: $D_{\vec{v}} f(a,b) = \nabla f(a,b) \cdot \vec{v}$

Meaning: $\nabla f(a,b)$ points in direction of fastest increase of f at (a,b). $|\nabla f(a,b)|$ is the rate of that increase.

Key Props: ① ∇f is at right angles to level sets.
② $\nabla f = \vec{0}$ at local min/max.

①



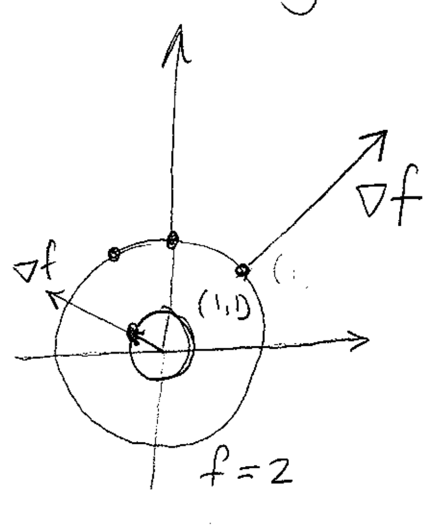
$\nabla f(a,b)$ is at right angles to the level set of f containing (a,b)

Ex: $f(x,y) = x^2 + y^2$

$\nabla f = (2x, 2y)$

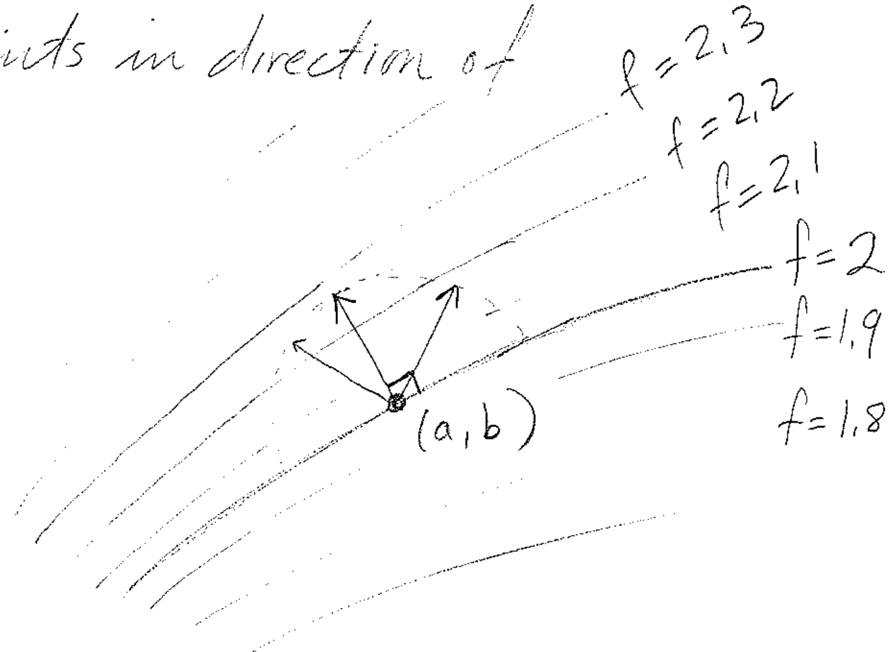
$f(1,1) = 2$

$\nabla f(1,1) = (2,2)$

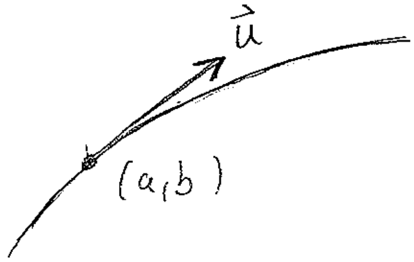


Reason 1: Gradient points in direction of fastest increase of f.

To cross as many level sets as possible with a vector of fixed length, should go at right angles to level sets.



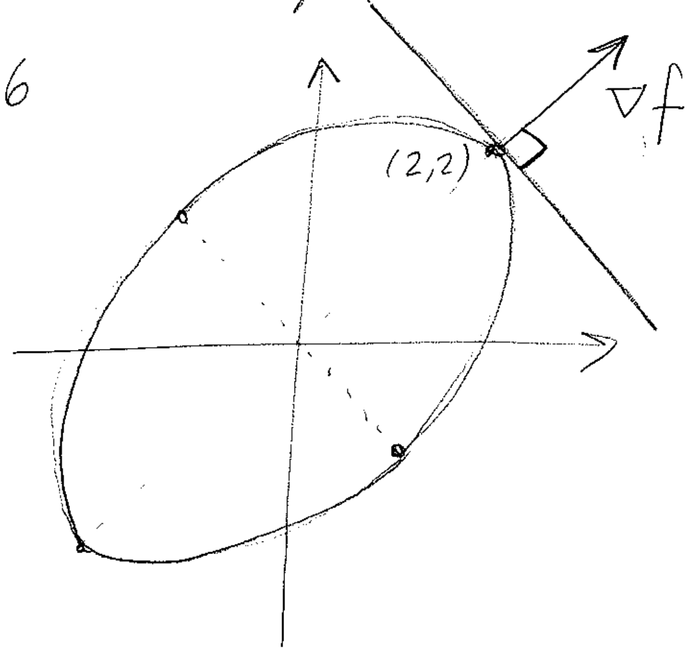
Reason 2: Suppose $\vec{u}$ is tangent to the level set at (a,b).



Since f is const on the level set, have $0 = D_{\vec{u}} f(a,b) = \nabla f(a,b) \cdot \vec{u}$.

Thus ∇f and $\vec{u}$ are $\perp$.

Ex: Find the tangent line to the ellipse C given by $5x^2 + 5y^2 - 6xy = 16$ at $(2,2)$.



View C as the level set $f = 16$ for

$$f(x,y) = 5x^2 + 5y^2 - 6xy$$

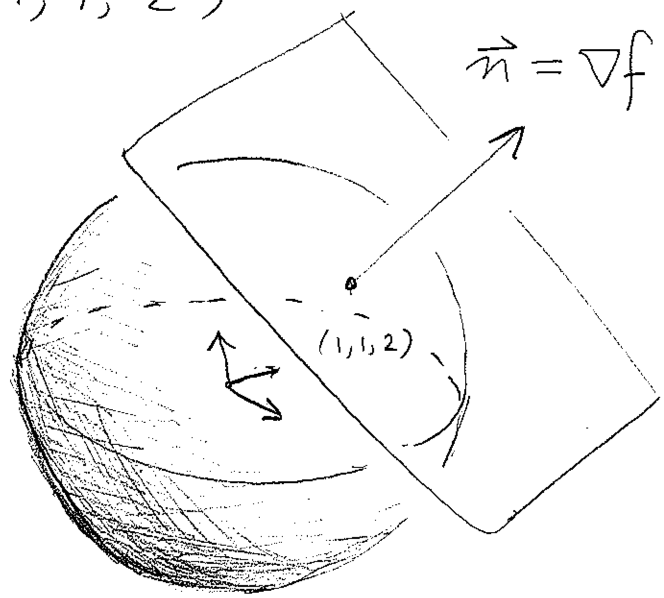
So $\nabla f = (10x - 6y, 10y - 6x)$ and

$$\nabla f(2,2) = (8, 8)$$

Thus the tangent line is $x + y = 4$.

Q: Find the tangent plane to the sphere

$$x^2 + y^2 + z^2 = 6 \quad \text{at} \quad (1, 1, 2)$$



A. Take $f(x,y,z) = x^2 + y^2 + z^2$

Then $\nabla f = (2x, 2y, 2z)$

and so take

$$\vec{n} = \nabla f(1,1,2) = (2,2,4)$$

as ∇f is $\perp$ to the level set (= sphere)

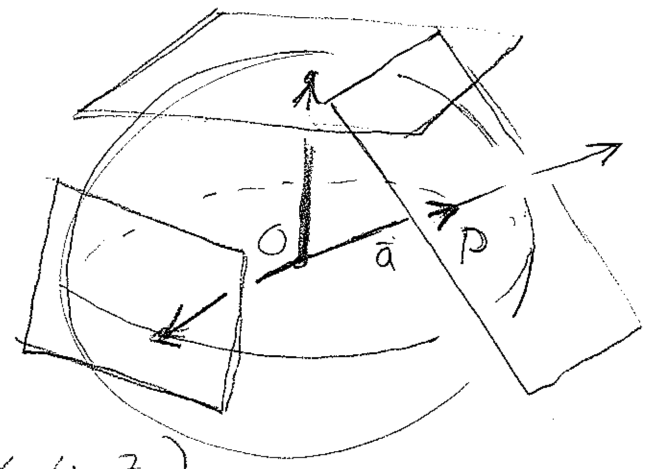
and so normal to tangent plane.

So eqn for plane is

$$2(x-1) + 2(y-1) + 4(z-2) = 0$$

$$\iff x + y + 2z = 6.$$

General Fact: If S is a sphere centered at $O=(0,0,0)$ and P a pt in S, then $\vec{a}=\overrightarrow{OP}$ is a normal vector to the tangent plane at P.



Reason: $f(x,y,z)=x^2+y^2+z^2$

$$\nabla f=(2x,2y,2z)=2(x,y,z)$$

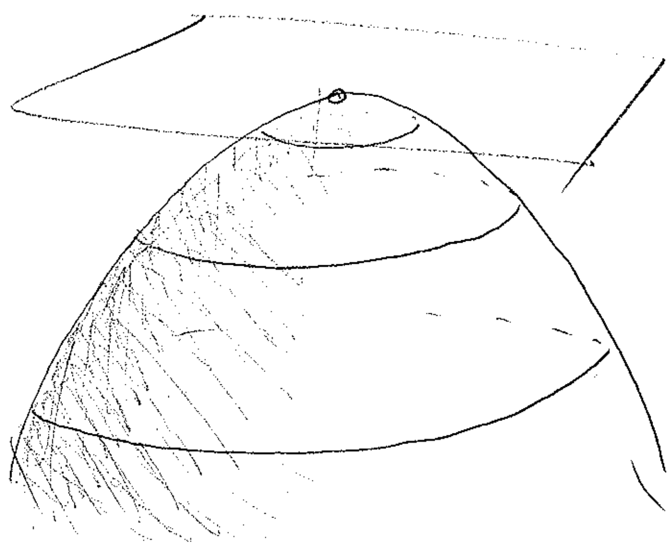
Optimization: $f: \mathbb{R}^2 \to \mathbb{R}$ want to find (a,b) where $f(a,b)$ is largest (or smallest).

Suppose (a,b) is where $f(a,b)$ is largest. What is $\nabla f(a,b)$?

A. $\nabla f(a,b)=\vec{0}$ [can't increase f at all]

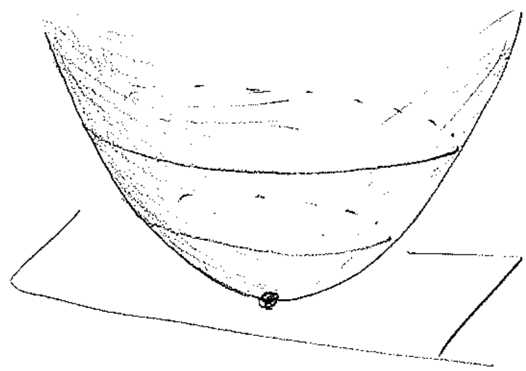
Ex: $f(x,y)=-x^2-y^2$

$$\nabla f=(-2x,-2y)$$



f takes its largest value at $(0,0)$ where
$\nabla f(0,0) = (0,0)$.

Suppose instead that $f(a,b)$ is the min value of f. What is $\nabla f(a,b)$?



Ex: $f(x,y) = x^2 + y^2$
min at $(0,0)$, where

$$\nabla f = (2x, 2y)$$

is $\vec{0}$. (Even though no mater which direction we go, f increases!)

In general $\nabla f(a,b) = \vec{0}$ at min values as well.

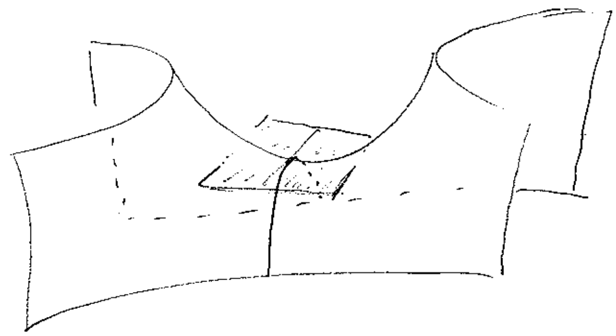
Note: $\nabla f(a,b) = \vec{0} \iff$ tangent plane is $=$ to xy-plane

Reason: If $f(a,b) = \vec{0}$ then $(z - f(a,b)) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$$= 0 + 0 = 0.$$

Another possibility: $f(x,y) = x^2 - y^2$

$\nabla f = (2x, -2y)$ which again is $\vec{0}$ at $(0,0)$



Q: How can we tell these apart? Need a 2^{nd} derivative test:

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \underbrace{\frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}}_{\text{Typically equal (Clairaut's Thm)}}$$

e.g. if both are cont.