Tuesday, August 30 $\ast\ast$ *Projections, distances, and planes.*

1. Let $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} - 1\mathbf{j}$.

   (a) Calculate $\text{proj}_{\mathbf{b}}\mathbf{a}$ and draw a picture of it together with $\mathbf{a}$ and $\mathbf{b}$.

   (b) The orthogonal complement of the vector $\mathbf{a}$ with respect to $\mathbf{b}$ is defined by

   $$\text{orth}_{\mathbf{b}}\mathbf{a} = \mathbf{a} - \text{proj}_{\mathbf{b}}\mathbf{a}.$$

   Calculate $\text{orth}_{\mathbf{b}}\mathbf{a}$ and draw two copies of it in your picture from part (a), one based at $\mathbf{0}$ and the other at $\text{proj}_{\mathbf{b}}\mathbf{a}$.

   (c) Check that $\text{orth}_{\mathbf{b}}\mathbf{a}$ calculated in (b) is orthogonal to $\text{proj}_{\mathbf{b}}\mathbf{a}$ calculated in (a).

   (d) Find the distance of the point $(1, 1)$ from the line $(x, y) = t(2, -1)$. Hint: relate this to your picture.

2. Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^n$. Use the definitions of $\text{proj}_{\mathbf{b}}\mathbf{a}$ and $\text{orth}_{\mathbf{b}}\mathbf{a}$ to show that $\text{orth}_{\mathbf{b}}\mathbf{a}$ is always orthogonal to $\text{proj}_{\mathbf{b}}\mathbf{a}$.

3. Find the distance between the point $P(3, 4, -1)$ and the line $\mathbf{l}(t) = (2, 3, -2) + t(1, -1, 1)$. Hint: Consider a vector starting at some point on the line and ending at P, and connect this to what you learned in Problem 1.

4. Consider the equation of the plane $x + 2y + 3z = 12$.

   (a) Find a normal vector to the plane. (Just look at the equation!)

   (b) Find where the x, y, and z-axes intersect the plane. Using this information, sketch the portion of the plane in the first octant where $x \geq 0, y \geq 0, z \geq 0$.

   (c) Using the points in part (b), find two non-parallel vectors that are parallel to the plane.

   (d) Using part (c) and the cross product, find another normal vector to the plane. Show that this vector is parallel to the vector from part (a).

   (e) Using the new normal vector and one of the points from (b), find an alternative equation for the plane. Compare this new equation to $x + 2y + 3z = 12$. How are these two equations related? Is it clear that they describe the same set of points (x, y, z) in $\mathbb{R}^3$?

5. *The Triangle Inequality.* Let $\mathbf{a}$ and $\mathbf{b}$ be any vectors in $\mathbb{R}^n$. The triangle inequality states that $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$.

   (a) Give a geometric interpretation of the triangle inequality. (E.g. draw a picture in $\mathbb{R}^2$ or $\mathbb{R}^3$ that represents this inequality.)

   (b) Use what we know about the dot product to explain why $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}|\,|\mathbf{b}|$. This is called the Cauchy-Schwarz inequality.

   (c) Use part (b) to justify the triangle inequality. Hint: Start with the fact that $|\mathbf{a} + \mathbf{b}|^2 = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b})$ and then use properties of the dot product and the Cauchy-Schwarz inequality to manipulate the right-hand side into looking like $|\mathbf{a}|^2 + |\mathbf{b}|^2$.