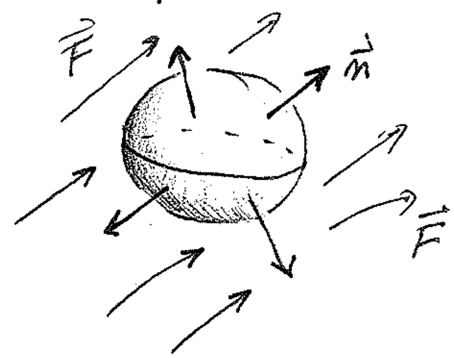
Lecture 39: Stokes Thm (16.8)

Last time: S surface in $\mathbb{R}^3$, $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ a vector field

$$\text{Flux} = \iint_S (\vec{F} \cdot \vec{n})\, dA$$ where $\vec{n}$ is a unit normal vector field.



Divergence Thm: D a region in $\mathbb{R}^3$, $\vec{F}: D \to \mathbb{R}^3$ a vector field. Then

$$\iint_{\partial D} (\vec{F} \cdot \vec{n})\, dA = \iiint_D \operatorname{div} \vec{F}\, dV$$

Integral Thms:

1-d:

$$f(b) - f(a) = \int_a^b f'(t)\, dt$$

2-d: Green's Thm

$\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\, dA$$

where $\vec{F} = (P, Q)$.

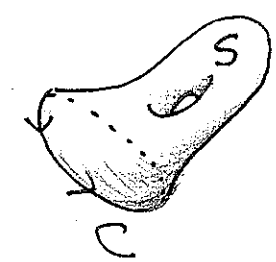
3-d: Divergence Thm.

1-d in 3-d:

$$f(B) - f(A) = \int_C \nabla f \cdot d\vec{r}$$

2-d in 3d: Stokes Thm

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$



$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\operatorname{curl} \vec{F}) \cdot \vec{n}\, dA$$

[Say something about how these are really the same...]

Curl: $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ a vector field with $F = (F_1, F_2, F_3)$

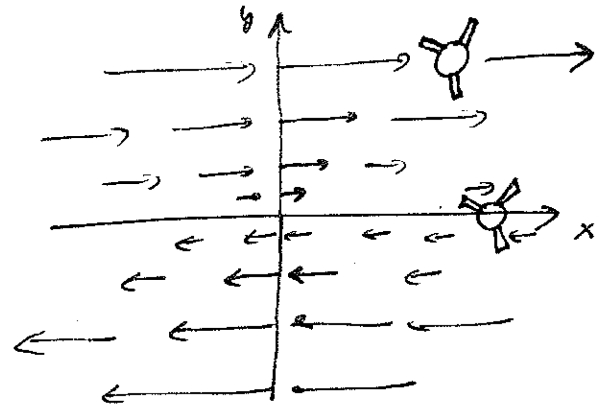
$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Another vector field.

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

Ex: $\vec{F} = (y, 0, 0)$

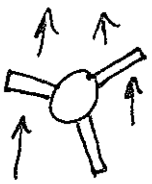
$\operatorname{curl} \vec{F} = (0, 0, -1)$



Q: What does curl measure?

depend only on (x,y).

First, consider $\vec{F} = (\vec{F}_1, \vec{F}_2, 0)$. [which is effectively a vector field on $\mathbb{R}^2$.] Place a small paddle wheel into the flow. As it moves along with the flow,



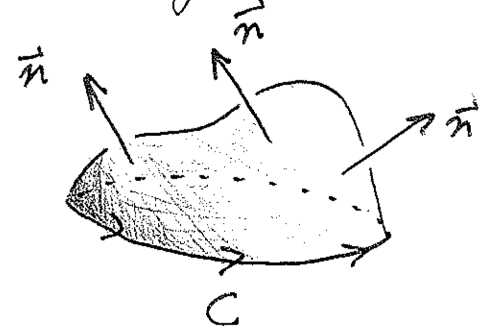
$|\operatorname{curl} \vec{F}|$ is the rate of rotation [precisely $2 \cdot$ (angular velocity)]. Also

$$\operatorname{curl} \vec{F} = \left(0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$ where is > 0 if rotation is anticlockwise.

[Will discuss the general meaning of curl $\vec{F}$ next time.]

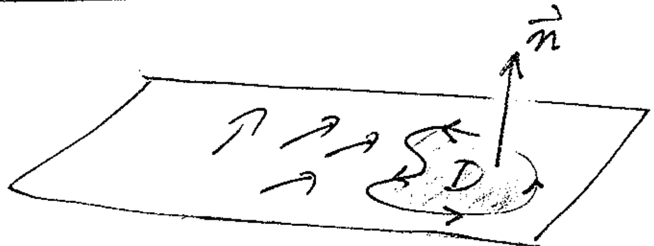
Stokes Thm: S surface in $\mathbb{R}^3$ with boundary curve C.

$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ a vector field.



$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\operatorname{curl} \vec{F}) \cdot \vec{n} \, dA$$

Relation to Green's Thm:



$\vec{F} = (\vec{F}_1, \vec{F}_2, 0)$

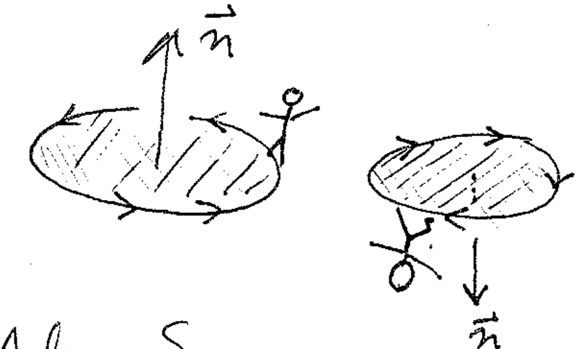
$\vec{n} = \vec{k}$

So

$$(\operatorname{curl} \vec{F}) \cdot \vec{n} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

C oriented so that S is to your left as you walk around C with head pointing in the normal direction.

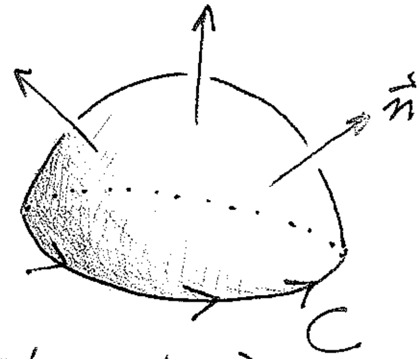


Also, S is orientable.

Example: S = upper unit hemisphere

$\vec{n}$ = outward normal

$\vec{F} = (y, xz, 1)$



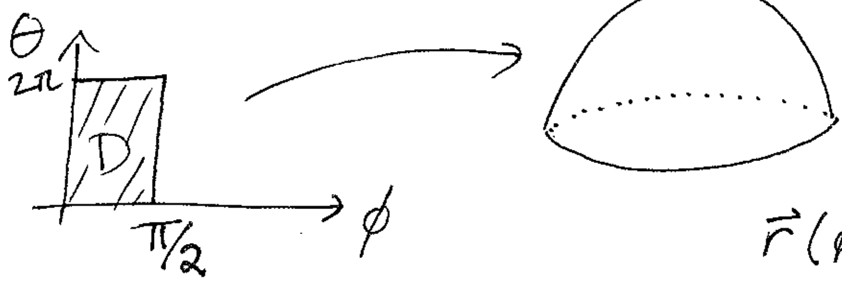
$\vec{r}(t) = (\cos t, \sin t, 0)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (\sin t, 0, 1) \cdot (-\sin t, \cos t, 0)\, dt$$

$$= \int_0^{2\pi} -\sin^2 t\, dt = \boxed{-\pi.}$$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & xz & 1 \end{vmatrix} = (-x, 0, z-1)$$

Parameterize S:



$dA = \sin\phi \, d\theta \, d\phi$

$\vec{r}(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$

$$\iint_S (\operatorname{curl} \vec{F}) \cdot \vec{n}\, dA = \iint_S (x, 0, z-1) \cdot (x, y, z)\; dA$$

$$= \iint_S -x^2 + z^2 - z\, dA = \int_0^{\pi/2}\int_0^{2\pi} (-\sin^2\phi\cos^2\theta + \cos^2\phi - \cos\phi)\sin\phi\, d\theta\, d\phi$$

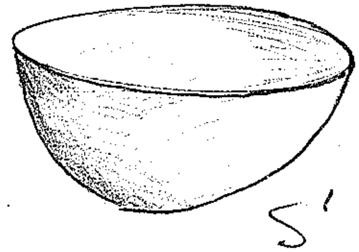
$$= \int_0^{\pi/2} \left(-\pi \sin^2\phi + 2\pi\cos^2\phi - 2\pi\cos\phi\right)\sin\phi \, d\phi$$

$$-\sin^2\phi = -1 + \cos^2\phi$$

$$= \pi \int_0^{\pi/2} \left(-1 + 3\cos^2\phi - 2\cos\phi\right)\sin\phi \, d\phi \qquad \begin{aligned} u &= -\cos\phi \\ du &= \sin\phi \, d\phi \end{aligned}$$

$$= \pi \int_{-1}^{0} (-1 + 3u^2 + 2u)\, du = \pi\left(-u + u^3 + u^2\right)\Big|_{-1}^{0} = \boxed{-\pi}$$

What about the lower hemisphere:



$$\iint_{S'} (\text{curl}\,\vec{F}) \cdot \vec{n} \, dA$$

$$= \int_{\pi/2}^{\pi}\int_0^{2\pi} (\text{same as before})\, d\theta\, d\phi = \pi\left(-u + u^3 + u^2\right)\Big|_{u=0}^{1} = \boxed{\pi}$$

But C is also the boundary of S', so shouldn't we get $\int_C \vec{F}\cdot d\vec{r} = -\pi$?

<u>Solution</u>: With outward normal, C gets oriented the other way by S'.