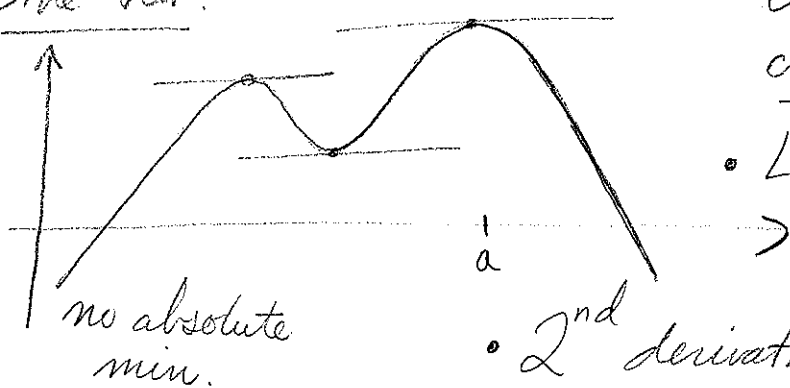


One Var:



- Min/max occur at critical pts where $f'(x) = 0$
- Local vs. Absolute extrema.

• 2nd derivative test: $\begin{cases} f''(a) < 0 \Rightarrow \text{max} \\ f''(a) > 0 \Rightarrow \text{min} \end{cases}$

[Need to do the same for $f: \mathbb{R}^n \rightarrow \mathbb{R}$, start with]

For $f: \mathbb{R}^2 \rightarrow \mathbb{R}$, (a,b) is a critical pt when

$\nabla f(a,b) = \vec{0}$

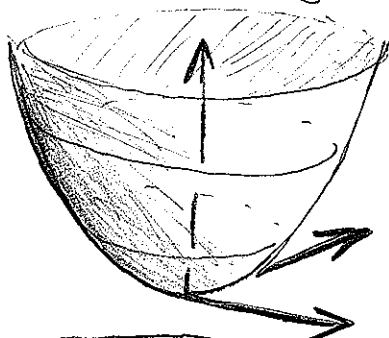
Every local extrema occurs at a critical pt.

[Today: 2nd derivate test for fns of two vars.]

Tale of 3 critical pts at $(0,0)$

$f(x,y) = x^2 + y^2$

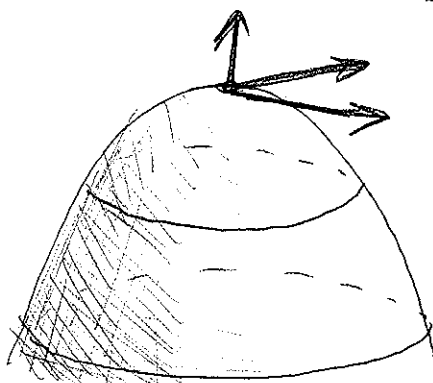
$\nabla f = (2x, 2y)$



Local Min

$f(x,y) = -x^2 - y^2$

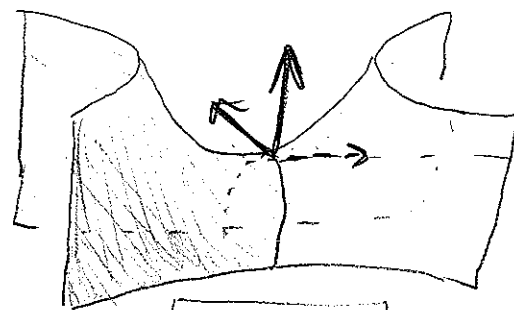
$\nabla f = (-2x, -2y)$



Local Max

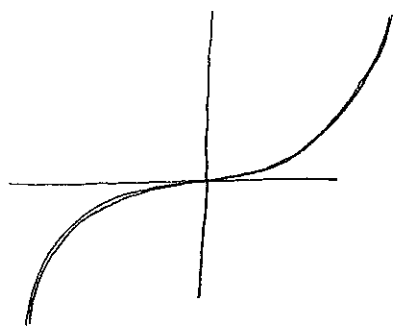
$f(x,y) = x^2 - y^2$

$\nabla f = (2x, -2y)$



Saddle

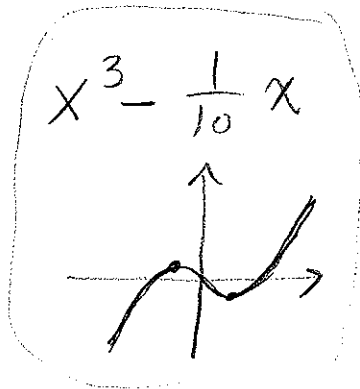
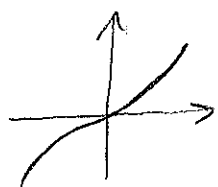
In one var, also a neither case, e.g. $f'(x) = x^3$



$$\left. \begin{aligned} f'(x) &= 3x^2 \\ f''(x) &= 6x \end{aligned} \right\} \text{both 0 at 0.}$$

This was pretty rare, because its not "stable".

E.g. $f(x) = x^3 + \frac{1}{10}x$ and $x^3 - \frac{1}{10}x$ don't have this issue.



However, saddles are stable.

One pt of view on the 2nd der. test in one var.

Taylor series: Near x_0 , usually have

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + E(h)$$

where $E(h)$ is really small, i.e. $\lim_{h \rightarrow 0} \frac{E(h)}{h^2} = 0$.

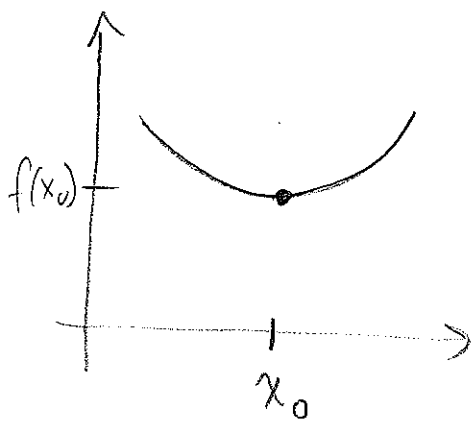
If x_0 is a critical pt, then

$$f(x_0+h) = f(x_0) + \frac{f''(x_0)}{2} h^2 + E(h)$$

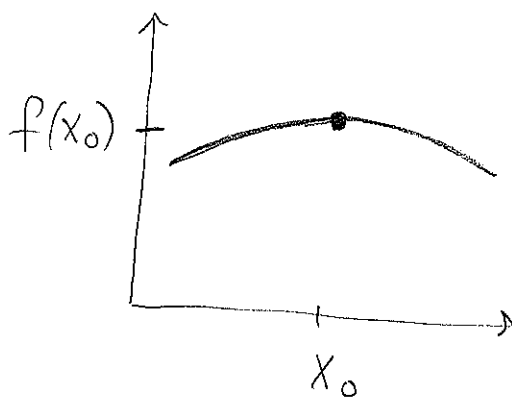
So near x_0 the graph of f looks like

$$f''(x_0) > 0$$

$$f''(x_0) < 0$$



local min



local max.

Taylor series for $f(x,y)$: For nice functions, have

$$f(x_0+h, y_0+k) = \underbrace{f(x_0, y_0) + f_x(x_0, y_0)h + f_y(x_0, y_0)k}_{\text{linear approx}} + ah^2 + bhk + ck^2 + \underbrace{E(h,k)}_{\text{smaller than other terms}}$$

next level of accuracy.

smaller than other terms.

Q: What are a, b, c ?

A: $a = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x_0, y_0)$, $b = \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$, $c = \frac{1}{2} \frac{\partial^2 f}{\partial y^2}(x_0, y_0)$

Reason: Take $x_0 = y_0 = 0$, $x = h, y = k$

$$f(x, y) \approx \underbrace{f(0, 0) + f_x(0, 0)x + f_y(0, 0)y + ax^2 + bxy + cy^2}_{g(x, y)}$$

①

$$f(0, 0) = g(0, 0)$$

②

$$f_x(0, 0) = g_x(0, 0) \text{ since } g_x(x, y) = f_x(0, 0) + 2ax + by$$

③ Want

$$f_{xx}(0, 0) = g_{xx}(0, 0)$$

as well. Since $g_{xx} = 2a$ this gives $a = \frac{1}{2} f_{xx}(0, 0)$.

Ex: Why Taylor series are so useful:

(45)

$$f(x,y) = \sin\left(\sqrt{1 + \frac{x^2}{2 + \cos y}} - e^{-xy}\right)$$

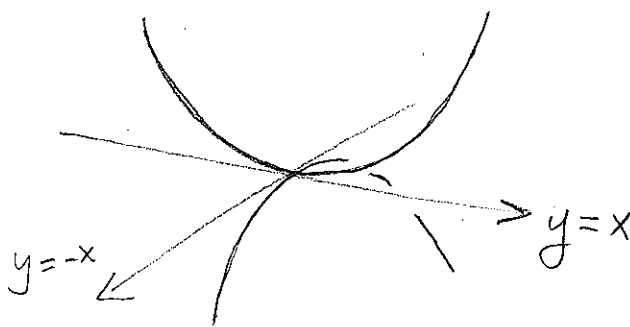
Now $(0,0)$ is a crit pt of this mess, but is it a max? Taylor series is:

$$f(x,y) = \frac{1}{6}x^2 + xy + E(x,y)$$

small compared to the other terms.

Look along lines:

$$f(x,x) = \frac{7}{6}x^2 + E(x,x)$$



$$f(x,-x) = \frac{1}{6}x^2 + x(-x) + E(x,-x) = -\frac{5}{6}x^2 + E(x,-x)$$

So f has a saddle at $(0,0)$.

2nd derivative test: Suppose (a,b) is a crit pt of f .

Set

$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

If $D > 0$ and $f_{xx}(a,b) > 0$ then (a,b) is a local min

If $D > 0$ and $f_{xx}(a,b) < 0$ then (a,b) is a local max

If $D < 0$ then (a,b) is a saddle.

[clg $D = 0$ or $f_{xx}(a,b) = 0$, break glass...]

Ex: $f(x,y) = x^2 + y^2$ $f_x = 2x$ $f_y = 2y$



At $(0,0)$ have

$$D = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0 \quad \text{and} \quad f_{xx}(0,0) = 2 > 0$$

so a min ✓

Ex: $f(x,y) = -x^2 - y^2$ $D = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0$ and $f_{xx}(0,0) < 0$

so a max ✓



Ex: $f = x^2 - y^2$ $D = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 < 0$ } Saddles at $(0,0)$

Ex: $f = xy$ $D = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0$

from worksheet.

Under the hood: Changing Coordinates, learn more about on Tuesday.

Higher derivs: Eigenvalues etc...