Lecture 11: The Chain Rule (14.5) and Directional Derivatives (14.6)

Last time: Chain Rule: ① $f: \mathbb{R}^2 \to \mathbb{R}$, $x, y: \mathbb{R} \to \mathbb{R}$

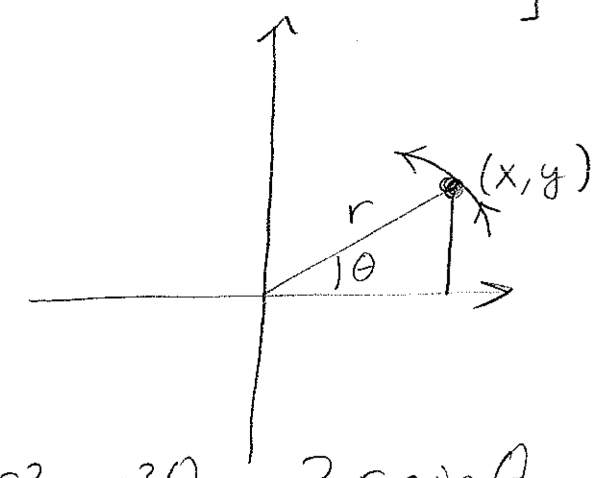
For $h(t) = f(x(t), y(t))$, we have

$$h'(t) = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t))\, y'(t)$$

② $z = f(x,y)$ with $x = x(t)$ and $y = y(t)$. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

[What if x and y themselves depend on more that one variable?]



Ex: $z = f(x,y) = x^2 + 3y$

$$x(r,\theta) = r\cos\theta$$
$$y(r,\theta) = r\sin\theta$$

So $z(r,\theta) = f(x(r,\theta), y(r,\theta)) = r^2\cos^2\theta + 3r\sin\theta$

Chain Rule: $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial \theta}$

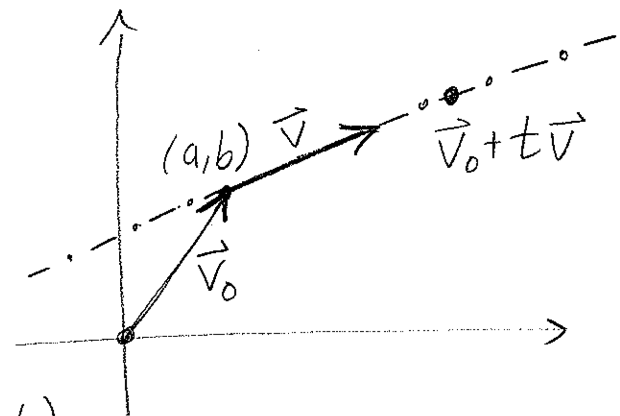
$$= (2x)(-r\sin\theta) + (3)\cdot(r\cos\theta)$$
$$= (2r\cos\theta)(-r\sin\theta) + 3r\cos\theta = -2r^2\sin\theta\cos\theta + 3r\cos\theta$$

[Comments: More Vars; How to remember; other var names ok.]

Directional Derivatives (14.6)

[Already have ∂-derivatives, measuring rates of change in x and y directions. But there's nothing special about these.]

Pick at pt (a,b) in $\mathbb{R}^2$ and a vector $\vec{v}$.



The derivative of f in direction $\vec{v}$ at (a,b) is:

$D_{\vec{v}} f(a,b) =$ rate of change in f as we move in direction $\vec{v}$ from (a,b).

$$= \frac{d}{dt} \underbrace{f(\vec{v}_0 + t\vec{v})}_{\text{fn of one var}} \Big|_{t=0}$$

<u>Ex</u>: For $\vec{v} = \vec{i}$ have $D_{\vec{i}} f(a,b) = \frac{\partial f}{\partial x}(a,b)$

In general, compute using the Chain Rule:

Have $\vec{v}_0 + t\vec{v} = (a + tv_1, b + tv_2)$, so

$\vec{v}_0 = (a,b)$ $\qquad$ $f(\vec{v}_0 + t\vec{v}) = f(x,y)$ where

$\vec{v} = (v_1, v_2)$ $\qquad\qquad\qquad\qquad\qquad$ $x = a + tv_1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $y = b + tv_2$

Now

$$D_{\vec{v}} f(a,b) = \frac{df}{dt}(0) = \frac{\partial f}{\partial x}(x(0),y(0))\,x'(0) + \frac{\partial f}{\partial y}(x(0),y(0))\,y'(0)$$

$$= \frac{\partial f}{\partial x}(a,b)\,v_1 + \frac{\partial f}{\partial y}(a,b)\,v_2$$

Ex: $f(x,y) = x^2 + y^3$

$\vec{u} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j}$

Unit vector: Usually want to take dir. deriv. using unit vectors.

$$D_{\vec{u}} f(2,1) =$$

$$\frac{\partial f}{\partial x}(2,1) \cdot \frac{1}{\sqrt{2}} + \frac{\partial f}{\partial y}(2,1)\left(-\frac{1}{\sqrt{2}}\right) = 4 \cdot \frac{1}{\sqrt{2}} + 3\left(-\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}$$

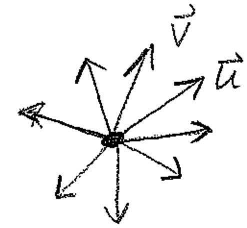
Gradient: $f: \mathbb{R}^2 \to \mathbb{R}$, define

$$\nabla f(a,b) = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right)$$

[Will give geom meaning shortly, but first notice.]

$$D_{\vec{v}} f(a,b) = \nabla f(a,b) \cdot \vec{v}$$

Q: In which direction does f increase the fastest?



[Why might we want to know this?]

Suppose $\vec{v}$ is a unit vector. Then

$$D_{\vec{v}}f(a,b) = \nabla f(a,b) \cdot \vec{v} = |\nabla f(a,b)| \cos\theta$$

To maximize, want $\theta = 0$, that is

$$\vec{V}_{max} = \frac{\nabla f(a,b)}{|\nabla f(a,b)|}.$$

Note also that

$$D_{\vec{V}_{max}} f = |\nabla f(a,b)|.$$

Summary: $\nabla f(a,b)$ points in the direction that f increases fastest. Its length is rate of said increase.

Ex: $f(x,y) = 1 - 4x^2 - y^2$

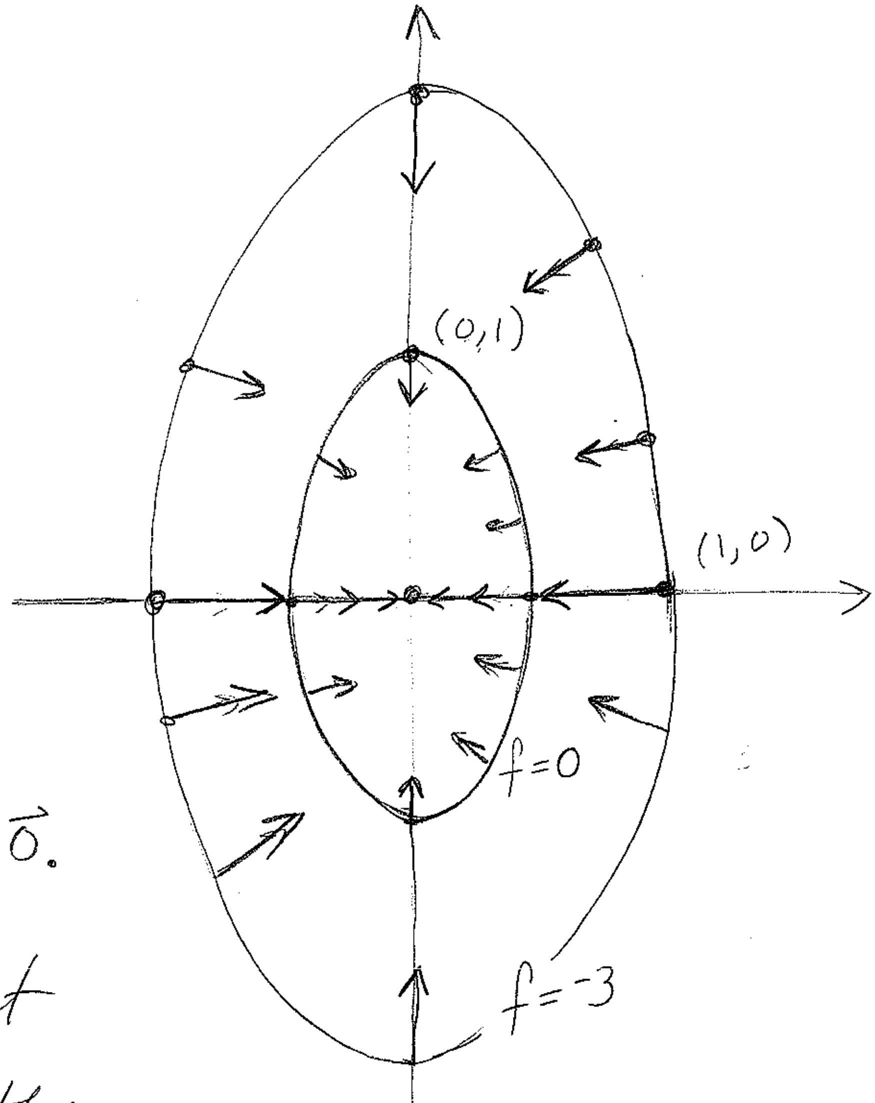
$$\nabla f = (-8x, -2y)$$

Level Sets: $f = 0 : 1 - 4x^2 - y^2 = 0 \iff 4x^2 + y^2 = 1$

$f = -3 : 1 - 4x^2 - y^2 = -3 \iff 4x^2 + y^2 = 4$

$\iff x^2 + \frac{y^2}{4} = 1$

$\nabla f(1,0) = (-8,0)$ $\quad \nabla f(0,1) = (0,-2)$ etc.

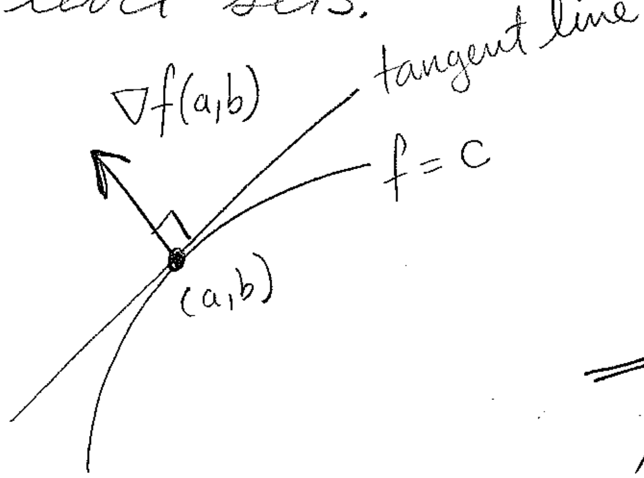


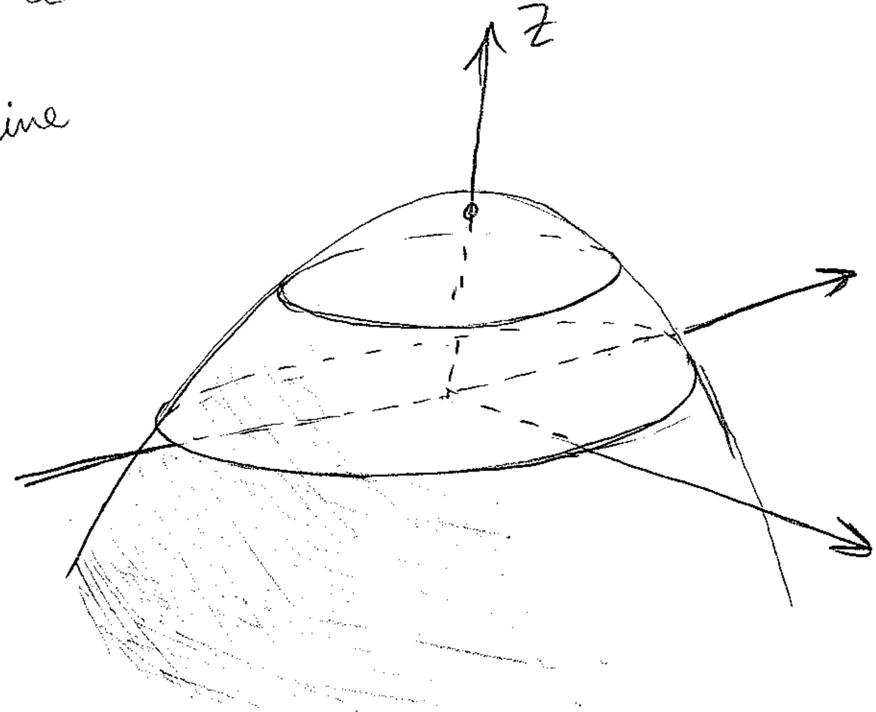
What is $\nabla f(0,0)$?

A: $\vec{0}$

Morals:

- A min/max of f can only occur when $\nabla f = \vec{0}$.
- ∇f is always at right angles to the level sets.

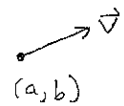




Lecture 12: Gradient (14.6)

Reminder: Exam Wed

Last time: $f: \mathbb{R}^2 \to \mathbb{R}$



$D_{\vec{v}} f(a,b) =$ rate f changes as we go in direction $\vec{v}$ starting at (a,b)

$$\nabla f(a,b) = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right)$$

Relationship: $D_{\vec{v}} f(a,b) = \vec{v} \cdot \nabla f(a,b)$

Meaning: $\nabla f(a,b)$ points in direction of fastest increase in f at (a,b). $|\nabla f(a,b)|$ is the rate of that increase.

Ex: $f(x,y) = 1 - 4x^2 - y^2$

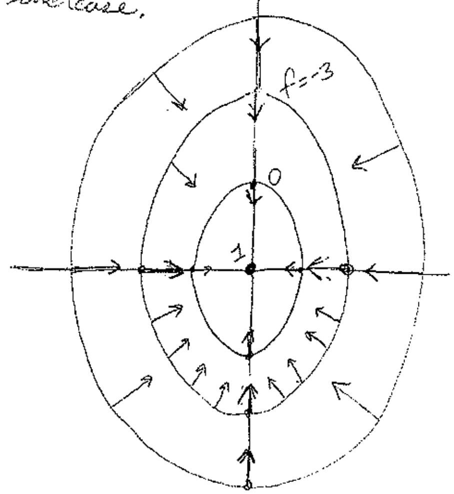
$\nabla f(x,y) = (-8x, -2y)$

Level sets:

$f = 1$: $(0,0)$

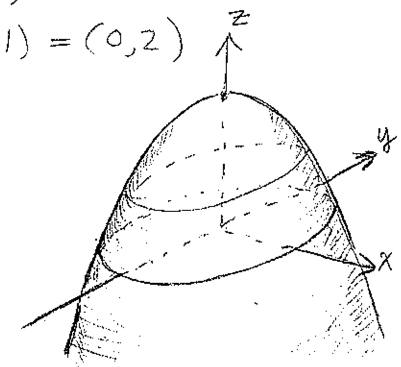
$f = 0$: $4x^2 + y^2 = 1$

$f = -3$: $4x^2 + y^2 = 4$

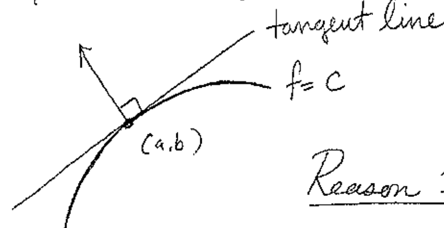


$\nabla f(1,0) = (-8,0)$ $\quad \nabla f(-1,0) = (8,0)$

$\nabla f(0,1) = (0,-2)$ $\quad \nabla f(0,-1) = (0,2)$

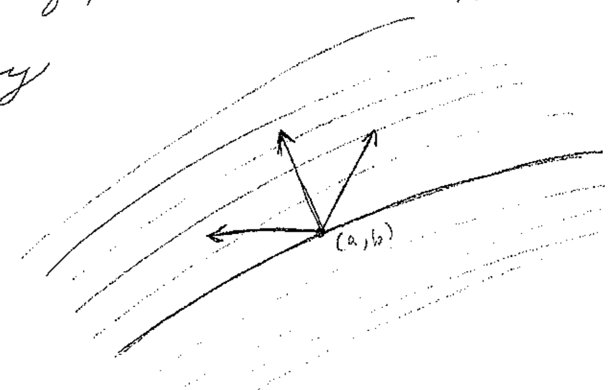


Key: $\nabla f(a,b)$ is at right angles to the level set of f containing (a,b).

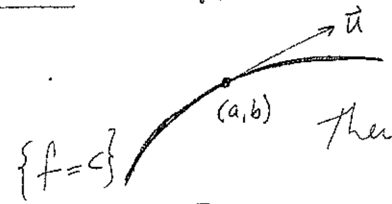


Reason 1: Gradient points in direction of fastest increase in f.

To cross as many level sets as possible with a vector of fixed length, should go at right angles to the level sets.



<u>Reason 2</u>: Suppose $\vec{u}$ is tangent to the level set at (a,b).



Then $D_{\vec{u}} f(a,b) = 0$

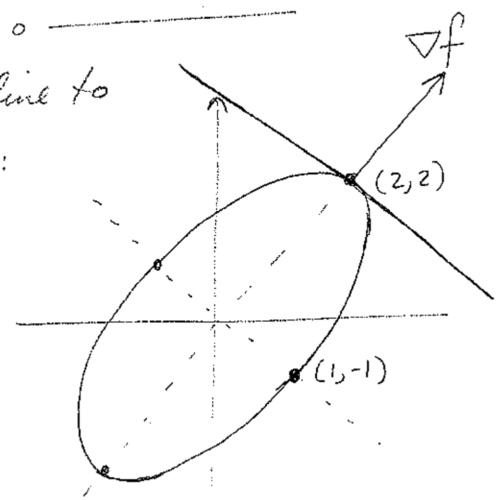
So $0 = \vec{u} \cdot \nabla f(a,b) \Rightarrow$

$\nabla f(a,b)$ is $\perp$ to $\vec{u}$.

<u>Ex</u>: Find the tangent line to the ellipse C given by:

$5x^2 + 5y^2 - 3xy = 8$

at $(2,2)$



<u>A</u>. View C as the level set $f = 8$ for

$f(x,y) = 5x^2 + 5y^2 - 3xy$

$\nabla f = (10x - 3y,\ 10y - 3x)$

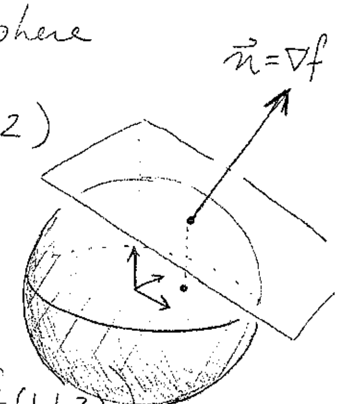
So $\nabla f(2,2) = (14, 14)$

and the line is: $x + y = 4$.

[Also works for more variables].

<u>Q</u>: Find the tangent to the sphere

$x^2 + y^2 + z^2 = 6$ at $(1,1,2)$



<u>A</u>. Take $f(x,y,z) = x^2 + y^2 + z^2$.

Then $\nabla f(1,1,2)$

$$= \left(\frac{\partial f}{\partial x}(1,1,2),\ \frac{\partial f}{\partial y}(1,1,2),\ \frac{\partial f}{\partial z}(1,1,2) \right)$$

$= (2, 2, 4)$ is at $\perp$ with the level set (= sphere)

and so is normal to the tangent plane.