Lecture 10: Chain Rule (Sect 14.5)

Last time: $f: \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a,b) if

$$f(a+h, b+k) = f(a,b) + \frac{\partial f}{\partial x}(a,b)\,h + \frac{\partial f}{\partial y}(a,b)\,k + E(h,k)$$

where $\displaystyle\lim_{(h,k)\to(0,0)} \frac{E(h,k)}{\sqrt{h^2+k^2}} = 0.$

Thm: If $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are cont near (a,b) then f is differentiable at (a,b).

Ex: $f(x,y) = -x^2 - y^2$ $\quad f_x = -2x$, $f_y = -2y$

$\Rightarrow$ f is diff everywhere.

Alt. notation: $\Delta x = h = x - a$ [Read in words.]

$\Delta y = k = y - b$

approximately

$$\Delta f = f(x,y) - f(a,b) \approx f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$$

[You all know]

$$\frac{d}{dt}\sin(t^2) = \cos(t^2)\,2t$$

[From next pg $f(t) = \sin(t)$ $g(t) = t^2$]

Setup: $f, g : \mathbb{R} \to \mathbb{R}$ then consider the composition $h = f \circ g$, that is $h(t) = f(g(t))$.

Chain Rule: $h'(t) = f'(g(t))\, g'(t)$

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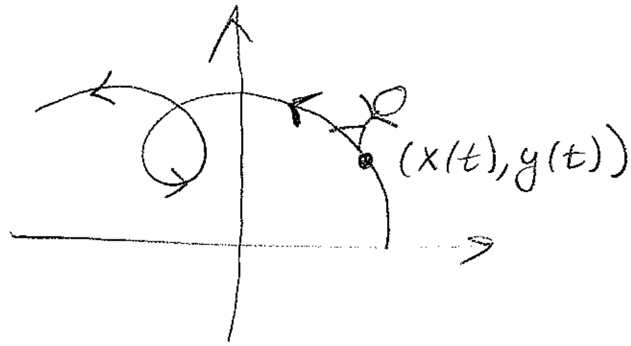
① $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

② parameterized curve $(x(t), y(t))$



③ Compose them

$$h(t) = f(x(t), y(t)) \quad \text{so } h : \mathbb{R} \longrightarrow \mathbb{R}.$$

Ex: f = temperature as a fn of position
h = temp as a fn of time.

Q: What is $h'(t)$? [In terms of der of x, y, + part of h]

That is, we want to find the linear approx

$$h(t+\Delta t) = h(t) + h'(t)\Delta t + E(\Delta t)$$

small compared to Δt

Know

$$x(t+\Delta t) = x(t) + x'(t)\Delta t + E_1(\Delta t)$$
$$y(t+\Delta t) = y(t) + y'(t)\Delta t + E_2(\Delta t)$$

and

$$f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x,y)\Delta x + f_y(x,y)\Delta y$$

So plug in and get

$$h(t+\Delta t) = f(x(t+\Delta t), y(t+\Delta t))$$

$$= f(\underbrace{x(t)}_{x} + \underbrace{x'(t)\Delta t + E_1(\Delta t)}_{\Delta x}, \underbrace{y(t)}_{y} + \underbrace{y'(t)\Delta t + E_2(\Delta t)}_{\Delta y})$$

$$\approx f(x(t),y(t)) + f_x(x(t),y(t))(x'(t)\Delta t + E_1(t))$$
$$+ f_y(x(t),y(t))(y'(t)\Delta t + E_2(t))$$

$$\approx h(t) + \big(f_x(x(t),y(t))x'(t) + f_y(x(t),y(t))y'(t)\big)\Delta t$$

where I've thrown away the terms with $E_1(t)$, $E_2(t)$.

<u>Chain Rule:</u> $h(t) = f(x(t), y(t))$

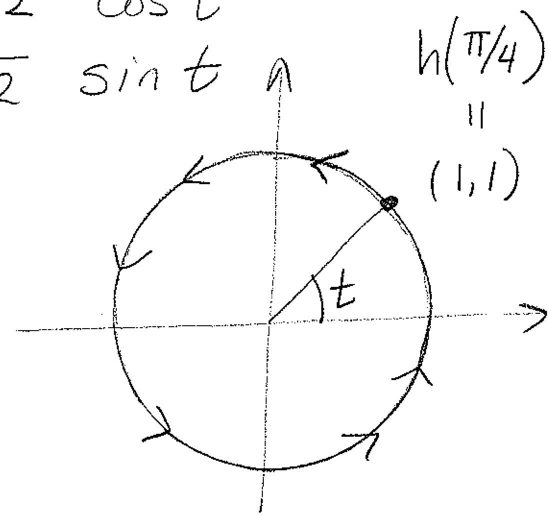
$$h'(t) = f_x(x(t),y(t))\,x'(t) + f_y(x(t),y(t))\,y'(t).$$

[Read in words.]

Ex: $f(x,y) = (x+y^2)^2$ $\quad x(t) = \sqrt{2}\cos t$

$y(t) = \sqrt{2}\sin t$

$h(t) = f(x(t), y(t))$



Find $h'(\pi/4)$: $x(\pi/4) = y(\pi/4) = 1$

$x'(t) = -\sqrt{2}\sin t \qquad x'(\pi/4) = -1$

$y'(t) = \sqrt{2}\cos t \qquad y'(\pi/4) = 1$

$f_x = 2(x+y^2) \qquad f_y = 2(x+y^2)\cdot 2y = 4y(x+y^2)$

So:

$$h'(\pi/4) = f_x(x(\pi/4), y(\pi/4))\, x'(\pi/4) + f_y(x(\pi/4), y(\pi/4))\, y'(\pi/4)$$

$$= f_x(1,1)(-1) + f_y(1,1)\cdot(1)$$

$$= 4(-1) + 8\cdot 1 = 4$$

Note: $h(t) = (2\cos t + 2\sin^2 t)^2$

so you can double check this directly in this case.

Alternate point of view: [Book likes]

$f, g : \mathbb{R} \to \mathbb{R}$ and $h(t) = f(g(t))$

Set $y \doteq f(x)$ and $x = g(t)$ so that y can be viewed as a function of t.

Then

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$ "Cancel the dx's."

Compare

$$\frac{dh}{dt}(t) = \frac{df}{dx}(g(t)) \cdot \frac{dg}{dt}(t) = f'(g(t)) \cdot g'(t)$$

Now suppose $h(t) = f(x(t), y(t))$.

Chain Rule:

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$ "Cancel ∂x with dx."

Sometimes write $z = f(x,y)$ and $x = x(t)$, $y = y(t)$ so that z becomes a function of t.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Also work for $f: \mathbb{R}^n \to \mathbb{R}$ for $n > 2$.

<u>Ex</u>: $W = f(x,y,z) = x^2 + yz$

$$x(t) = t$$
$$y(t) = t^2$$
$$z(t) = 1 - t$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$= 2x \cdot 1 + z \cdot 2t + y(-1)$$

$$= 2(t) \cdot 1 + (1-t)2t + t^2(-1)$$

$$= 2t + 2t - 2t^2 - t^2 = 4t - 3t^2$$

<u>Check</u>: $w(t) = t^2 + t^2(1-t) = 2t^2 - t^3$

$$w'(t) = 4t - 3t^2 \quad \checkmark$$