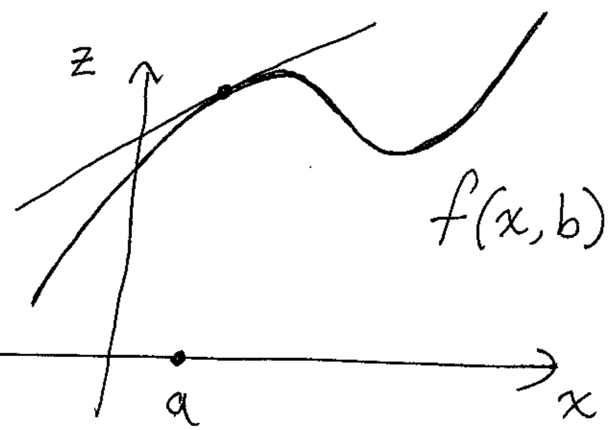
Lecture 8: Partial Derivatives + Applications (14.3 + 14.4)

Last time: $f: \mathbb{R}^2 \to \mathbb{R}$

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$$

= rate f changes as we move in x direction from (a,b).

= slope of line in slice



Ex: Just take other vars as consts.

$$\frac{\partial}{\partial x}(x^3y + xy) = 3x^2y + y$$

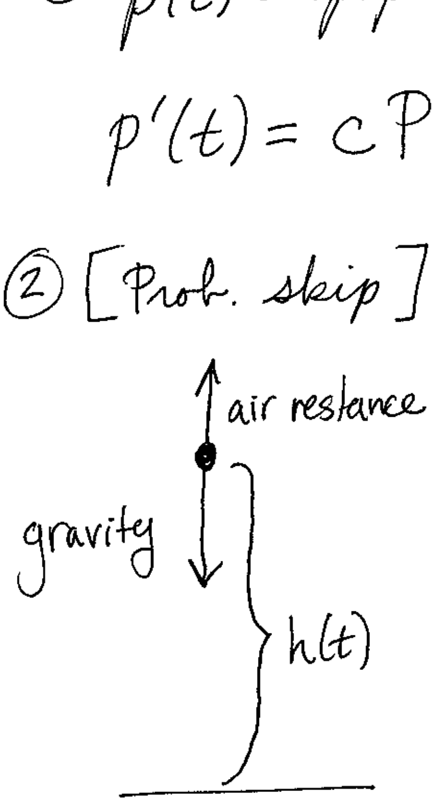
All partial ders together will play the same role as the derivative for one-var calc (e.g. min/max, tangent planes, Taylor series.)

O.D.E: Ordinary Differential Equations.

① $p(t) =$ population at time t

$$p'(t) = cP(t) \implies P(t) = P_0 e^{ct}$$

② [Prob. skip]

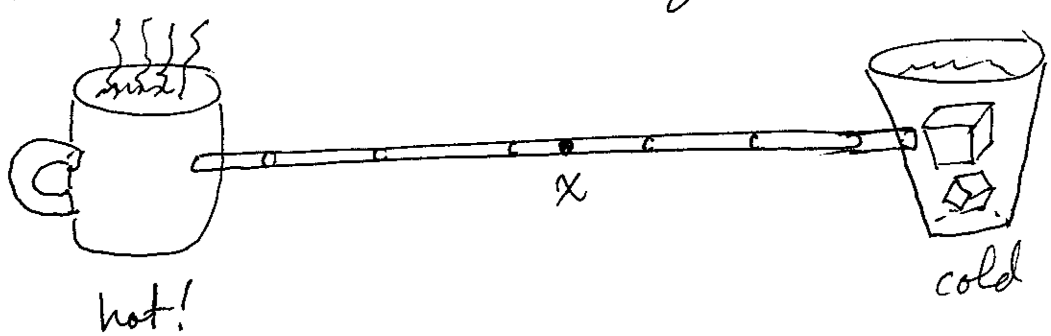


$$h''(t) = -g - ah'(t)$$

$$h(t) = -\frac{1}{a^2}\left(agt + (av_0 + g)e^{-at}\right)$$

[Subject of Math 285/286.]

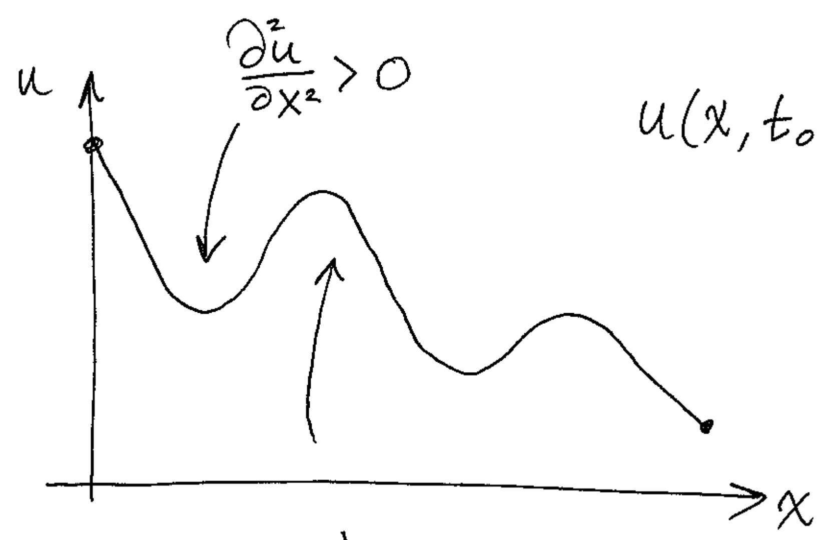
PDE. Partial Differential Equations.



$u(x,t) =$ temp of rod at pos x and time t.

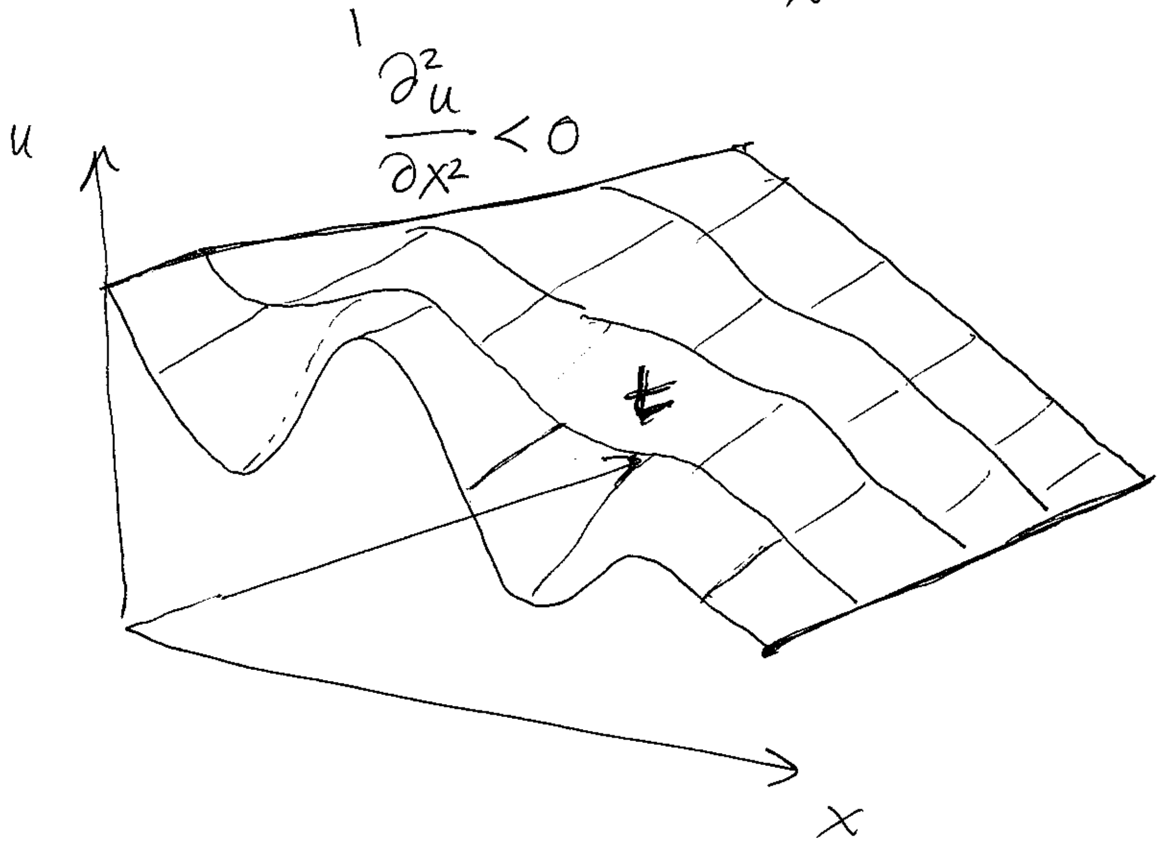
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x} u\right) \quad \text{[Heat Equation.]}$$

Comes from Newton's Law of Cooling: Heat flow is prop to $-\frac{\partial u}{\partial x}$

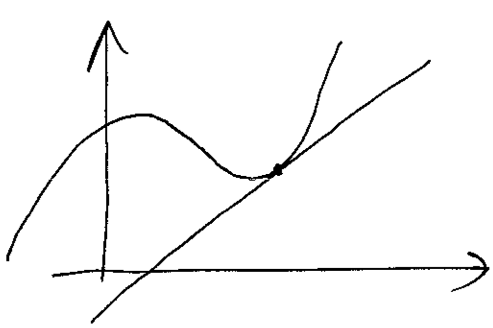


$$u(x, t_0) = v(x)$$

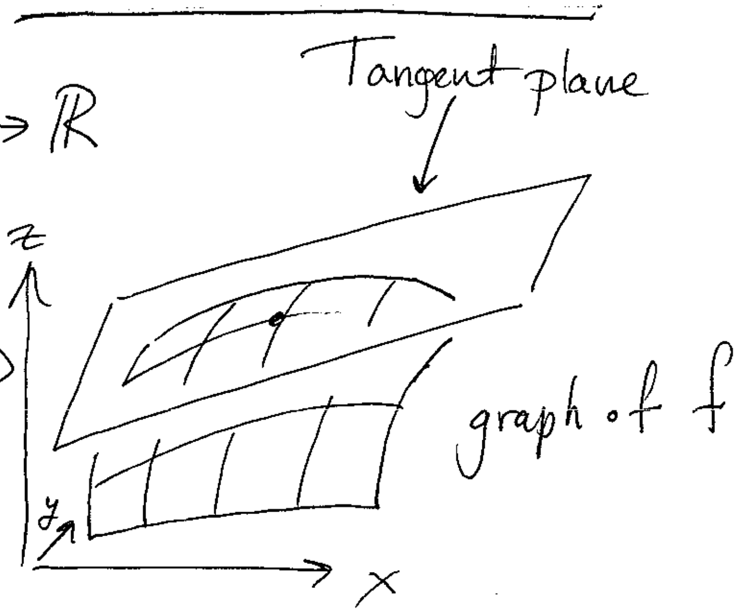
$$\frac{\partial^2 u}{\partial x^2}(x, t_0) = v''(x)$$



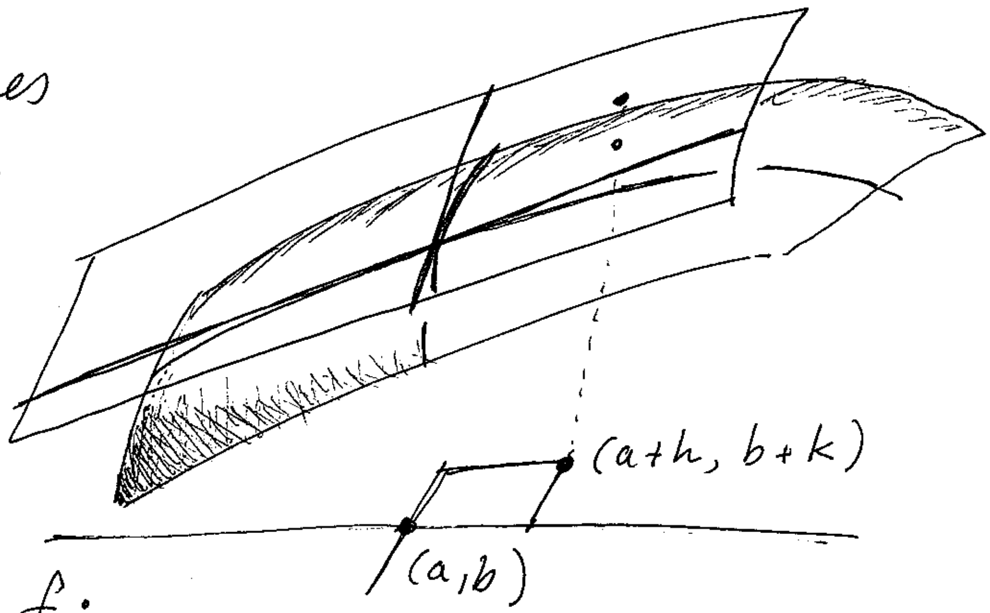
<u>Tangent plane</u>: $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$



$\Longrightarrow$



The tangent plane contains the tangent lines in the x and y slices:



To find the formula, approx. f:

$$f(a+h,b+k) = f(a,b) + \frac{\partial f}{\partial x}(a,b)\,h + \frac{\partial f}{\partial y}(a,b)\,k + E(h,k)$$

Say f is <u>differentiable</u> at (a,b) if

$$\lim_{(h,k)\to\vec{0}} \frac{|E(h,k)|}{\sqrt{h^2+k^2}} = 0.$$

Means that there is a tangent plane at $(a,b,f(a,b))$. Rewrite

$$f(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) + \text{Error}$$

So tangent plane is given by

$$z - f(a,b) = \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

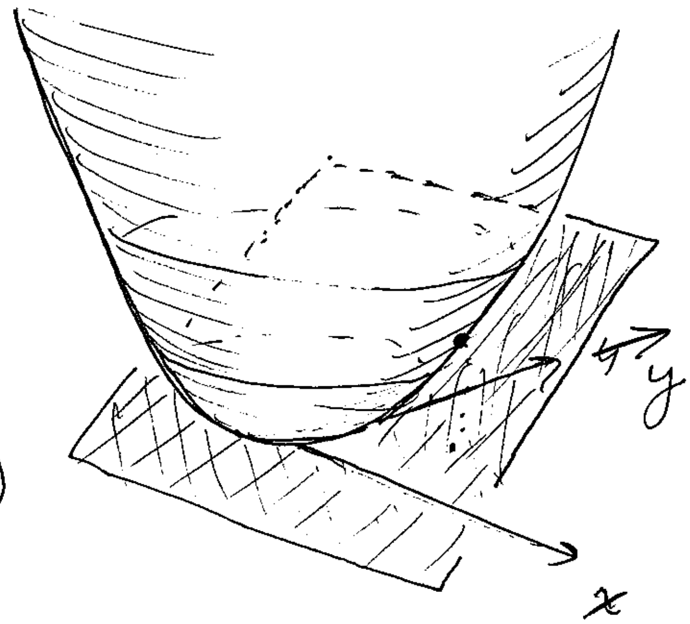
Ex: $f(x,y) = x^2 + y^2$

$$\frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = 2y$$

Tangent plane $(x,y) = (1,1)$

$$z - 2 = 2(x-1) + 2(y-1)$$

$$z = 2x + 2y - 2$$



Note: Just because $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (a,b) does not mean f is different. at (a,b).

Ex: $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$

$\frac{\partial f}{\partial x}(0,0) = 0$ since $\lim\limits_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim\limits_{h \to 0} \frac{0}{h} = 0$

$\frac{\partial f}{\partial y}(0,0) = 0$ But: f isn't continuous at $(0,0)$

as $\lim\limits_{(x,y) \to (0,0)} \frac{2xy}{x^2+y^2}$ does not exist.

This runs into: [just as in one var.]

Thm: $f: \mathbb{R}^2 \to \mathbb{R}$. If f is differentiable at (a,b) then f is continuous at (a,b).

Reason: As f is diff at (a,b) have: $\lim_{(h,k)\to 0} \frac{E(h,k)}{\sqrt{h^2+k^2}} = 0$

$$f(a+h, b+k) = f(a,b) + \frac{\partial f}{\partial x}(a,b)\,h + \frac{\partial f}{\partial y}(a,b)\,k + E(k,h)$$

Thus as $(h,k) \to (0,0)$ have

$$f(a+h, b+k) \longrightarrow f(a,b) + 0 + 0 + 0$$

So $\lim_{(x,y)\to(0,0)} f(x,y) = f(a,b)$ and f is cont.

Thm: $f: \mathbb{R}^2 \to \mathbb{R}$. Suppose $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist near (a,b) and are continuous near (a,b). Then f is differentiable at (a,b)

Ex: $f(x,y) = xy^3 + xy + \sin(xy)$ is differentiable on all of $\mathbb{R}^2$.