Lecture 19: Vector fields (16.1 and 16.2)

Last time:

C - curve in $\mathbb{R}^3$

$f: \mathbb{R}^3 \to \mathbb{R}$

$$\int_C f\,ds = \int_a^b f(\vec{r}(t))|\vec{r}\,'(t)|\,dt$$

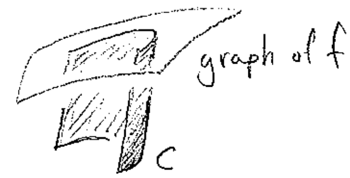
where $\vec{r}: [a,b] \to \mathbb{R}^3$ is a param. of C.

Meanings:

① $\text{Ave. of } f \text{ on } C = \frac{1}{\text{len}(C)} \int_C f\,ds$

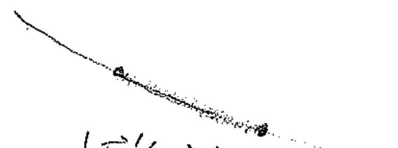
② $\text{Total mass} = \int_C f\,ds$ ← density fn

③ Area



Note: $ds = |\vec{r}\,'(t)|\,dt$ is called the "arc length" element.

Also $\int_C 1\,ds = \text{Length}(C)$.



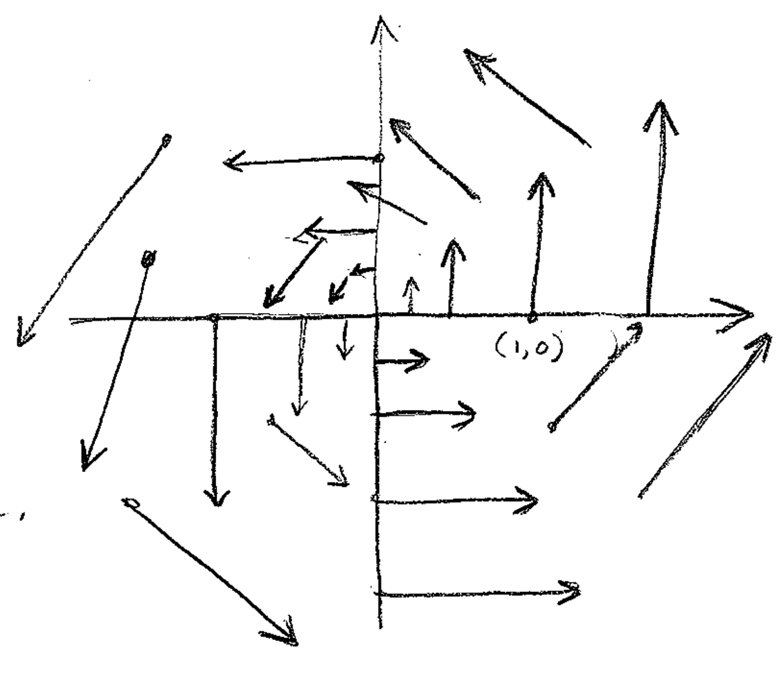
Vector Fields: (16.1 and 16.2)

For $\mathbb{R}^2$, a vector field is a function $\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$

Ex: $\vec{F}(x,y) = -y\vec{i} + x\vec{j}$

Uses:

- Wind speed / direction
- Fluid flow.
- Force magnitude / direction.
- Electric / magnetic fields.

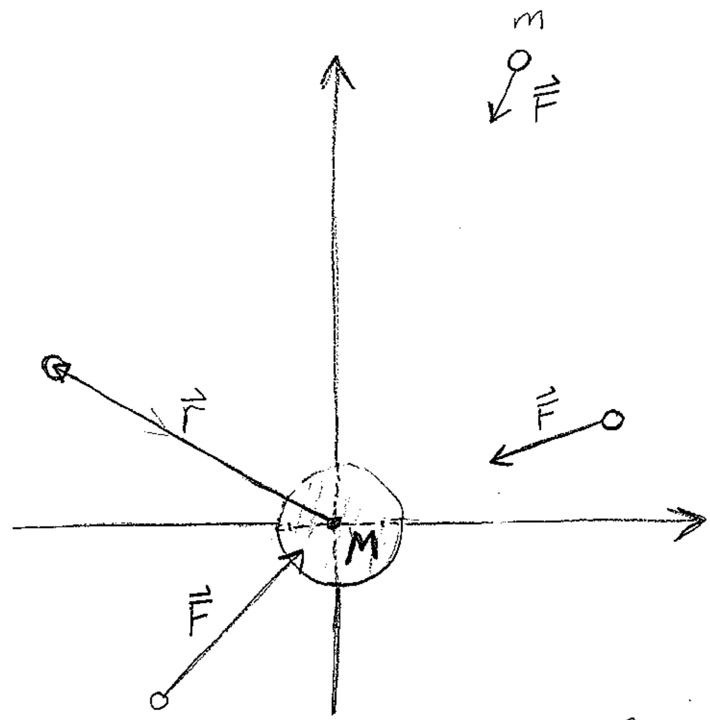


Ex: Gravity:

Large mass M at $(0,0)$.

Force $\vec{F}$ on small mass m

depends position.



$\vec{F}$ points in direction $-\vec{r}$

Newton's Law:

const

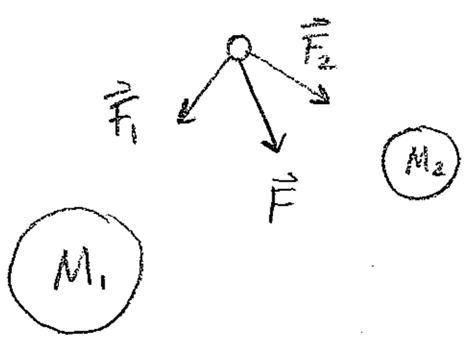
$$|\vec{F}| = \frac{MmG}{|\vec{r}|^2}$$

If $\vec{F} = -c\vec{r}$ then $|\vec{F}| = c|\vec{r}| \implies c = \frac{MmG}{|\vec{r}|^3}$

So

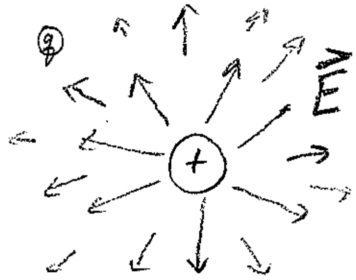
$$\vec{F} = -\frac{MmG}{|\vec{r}|^3}\vec{r}$$

For several bodies, add vector fields.



$$\vec{F}(x,y) = \vec{F_1}(x,y) + \vec{F_2}(x,y)$$

Ex: Electric field. Force on a charge q at (x,y) is



$$\vec{F} = \frac{E(x,y)}{q}$$

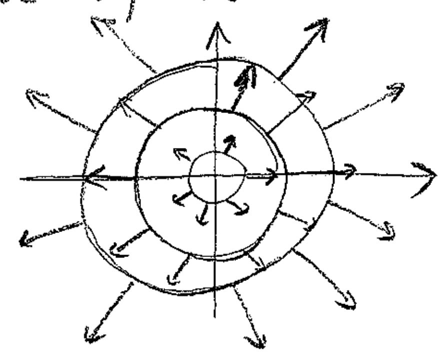
Where have we seen vector fields before?

A. The gradient!

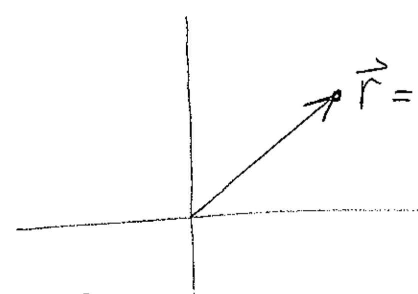
$f: \mathbb{R}^2 \to \mathbb{R}$ then $\nabla f: \mathbb{R}^2 \to \mathbb{R}^2$

Ex. $f(x,y) = x^2 + y^2$

$\nabla f = (2x, 2y)$



Ex: $f(x,y) = \dfrac{MmG}{\sqrt{x^2+y^2}} = \dfrac{MmG}{|\vec{r}|}$



$$\nabla f = MmG\left(-\tfrac{1}{2}(x^2+y^2)^{-3/2}\cdot(2x),\ \_\_\_\_\_\_\right)$$

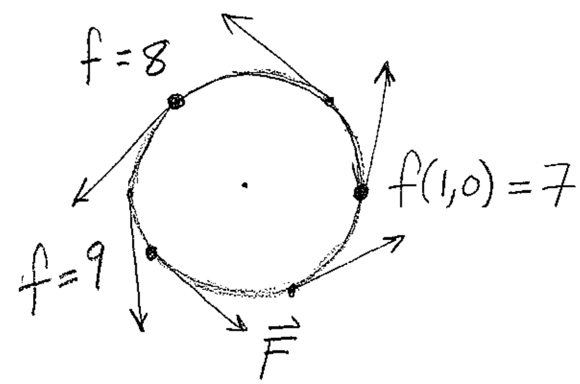
$$= MmG\left(\frac{x}{(x^2+y^2)^{3/2}},\ \frac{y}{(x^2+y^2)^{3/2}}\right) = \frac{MmG}{|\vec{r}|^3}\vec{r} = \vec{F}$$

Say that f is a <u>potential function</u> for $\vec{F}$, and that $\vec{F}$ is a <u>conservative vector field</u>.

Think potential energy.

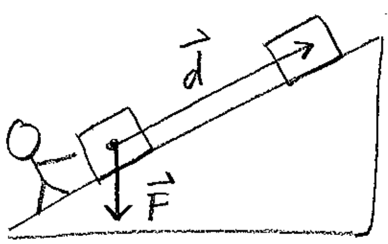
Q: Is $\vec{F} = y\vec{i} - x\vec{j}$ conservative?

A: No. Suppose $\nabla f = \vec{F}$. Since $\vec{F}$ is tangent to the unit circle, following the circle increases f. But going all the way around, we end up at back at $(1,0)$...



Integrating Vector Fields: (16.2)

Recall:

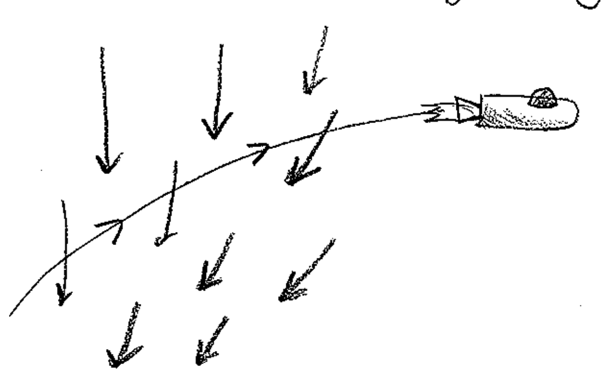


Work done by gravity

$$W = \vec{F} \cdot \vec{d}$$

[Assumes const force.]

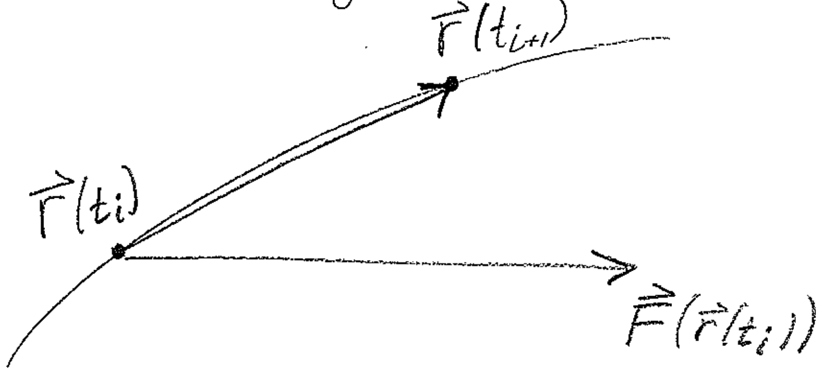
How much work does gravity do here?



Motion of ship $\vec{r}: \mathbb{R} \to \mathbb{R}^2$.

Force of grav = vector field $\vec{F}$.

Break into segments



$$\text{Work done here} \approx \vec{F}(\vec{r}(t_i)) \cdot \overbrace{(\vec{r}(t_{i+1}) - \vec{r}(t_i))}^{\approx \Delta t\, \vec{r}\,'(t_i)}$$

$$\approx \left(\vec{F}(\vec{r}(t_i)) \cdot \vec{r}\,'(t_i)\right) \Delta t$$

Sum up and take $\Delta t \to 0$ to get

$$\begin{matrix}\text{Total}\\ \text{Work}\end{matrix} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

<u>General Setup:</u> C a curve in $\mathbb{R}^n$

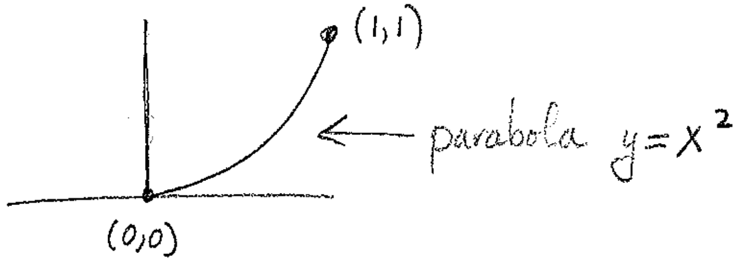
$\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ a vector field.

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

for any param $\vec{r}: [a,b] \to \mathbb{R}^n$.

[<u>Note</u>: Answer only depends on direction of param.]

Ex: $C =$



$\vec{r}(t) = (t, t^2)$ for $0 \leq t \leq 1$.

$\vec{r}\,'(t) = (1, 2t)$

$\vec{F} = (y, 1)$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^1 (t^2, 1) \cdot (1, 2t)\, dt$$

$$= \int_0^1 t^2 + 2t\, dt = \frac{t^3}{3} + t^2 \Big|_{t=0}^{t=1} = \frac{4}{3}.$$

Explain why this is consist. with the work interpret.