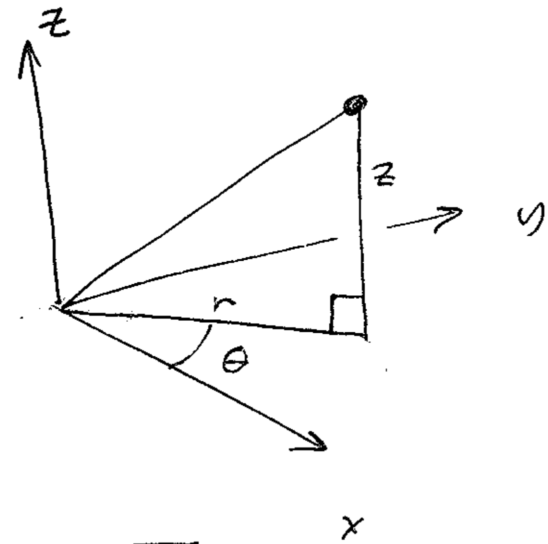
Lecture 29: Integrating in cylindrical and spherical coor. (15.7 and 8)

Last time: Cylindrical coordinates

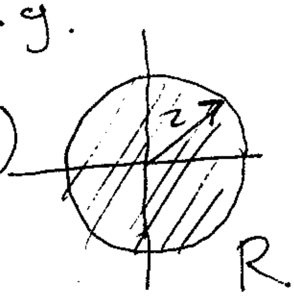
(r, θ, z) with $0 \le r$
$0 \le \theta \le 2\pi$



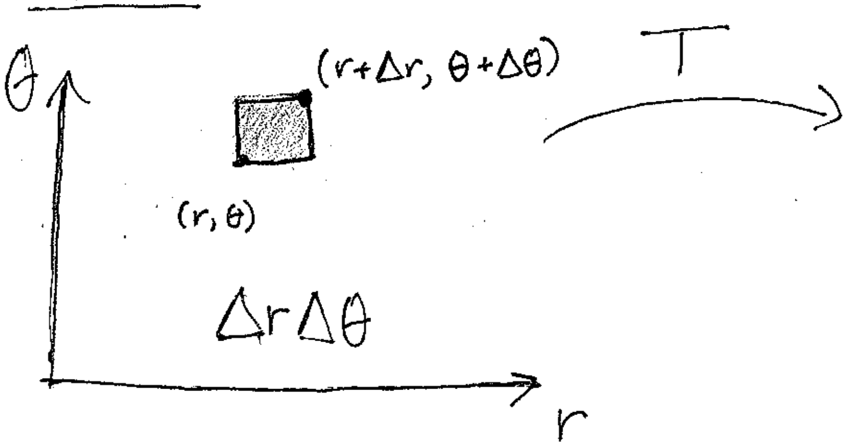
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

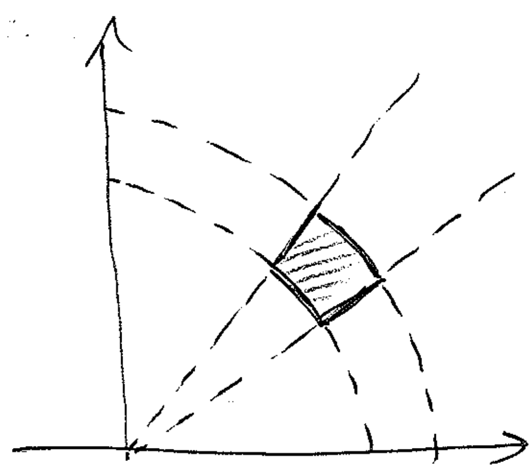
In polar coordinates $dA = r\,dr\,d\theta$, e.g.

$$\iint_R xy\,dA = \int_0^{2\pi}\int_0^{2} r^2\cos\theta\sin\theta\,(r\,dr\,d\theta)$$



Reason:

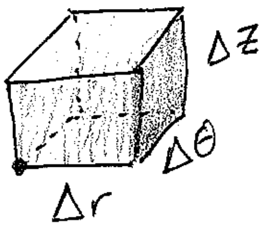


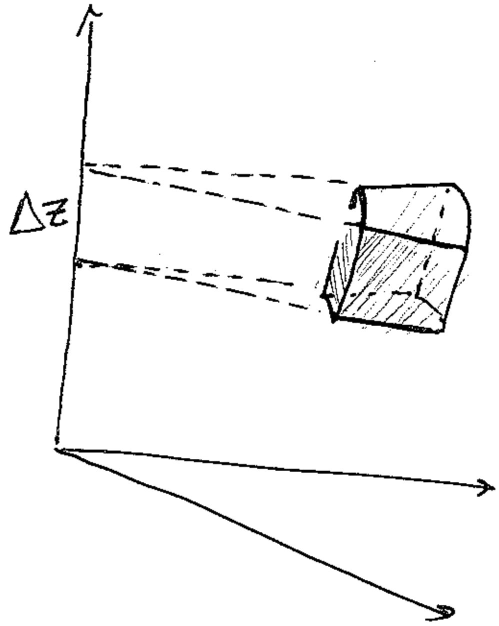


Area $\approx r\Delta r\Delta\theta$

$$T(r,\theta) = (r\cos\theta, r\sin\theta)$$

Cylindrical:





At right, volume is

$$\Delta z \cdot \text{Area} = \Delta z\,(r\,\Delta r\,\Delta\theta)$$

$$\Rightarrow dV = r\,dr\,d\theta\,dz$$

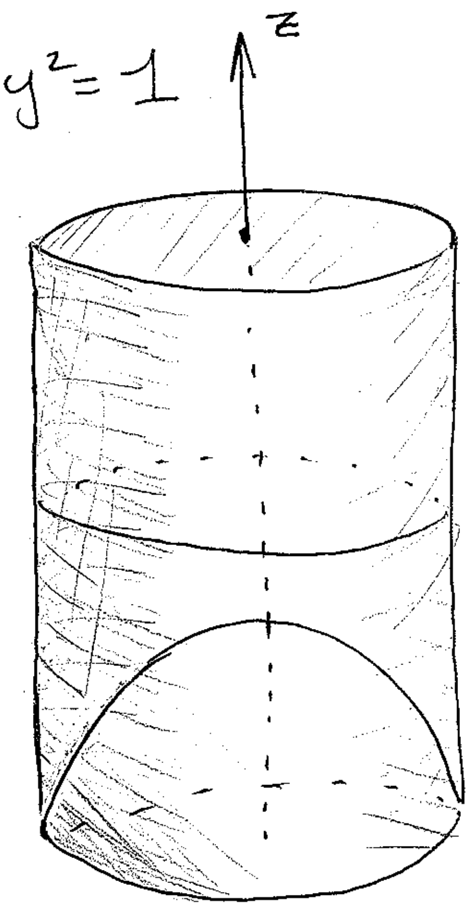
Ex: R inside cylinder $x^2+y^2=1$

below $z=4$

above $z = 1 - x^2 - y^2$
$= 1 - r^2$

$\rho(x,y,z) = \sqrt{x^2+y^2} = r$

Find total mass.

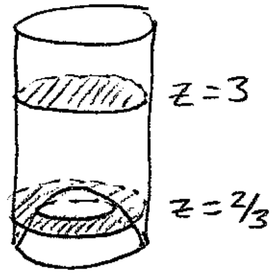


$$\iiint_R \rho\, dV$$

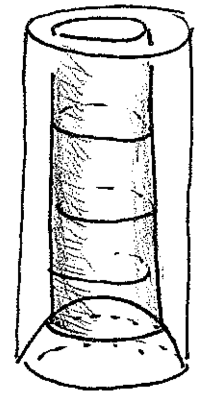
$x^2+y^2+z^2=1$

Q: How to slice?

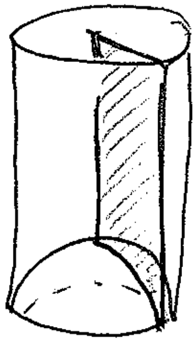
(a) by z:



(b) by r



(c)



by θ

Let's do r, since the book picks θ.

$$\int_0^1 \int_{1-r^2}^4 \int_0^{2\pi} \overbrace{r}^{\rho} \; \overbrace{r\, d\theta\, dz\, dr}^{dV}$$

$$= \int_0^1 \int_{1-r^2}^4 2\pi r^2 \, dz\, dr = \int_0^1 2\pi r^2 z \Big|_{z=1-r^2}^{4} dr$$

$$= 2\pi \int_0^1 r^2(4-(1-r^2))\, dr = 2\pi \int_0^1 3r^2 + r^4 \, dr$$

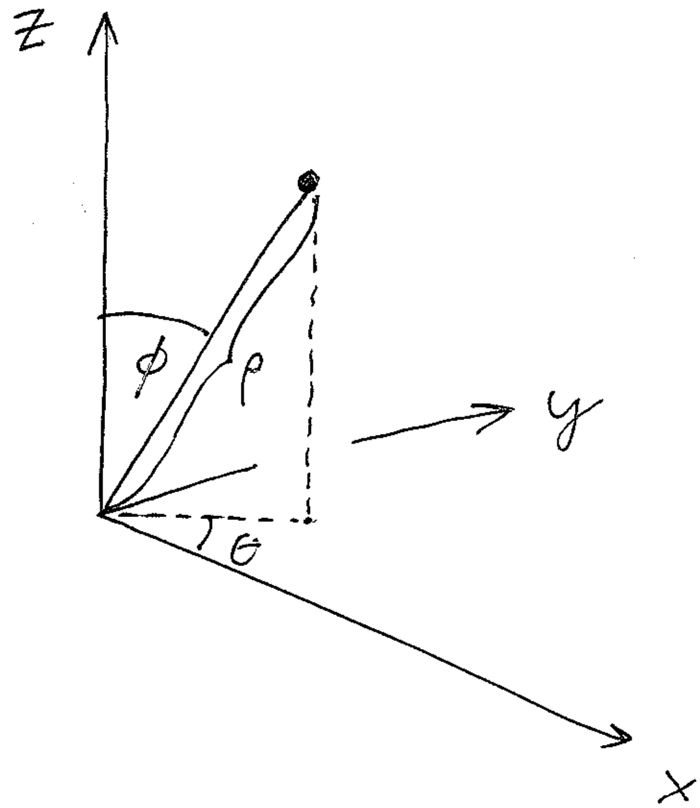
$$= 2\pi \left(r^3 + \frac{1}{5} r^5 \right) \Big|_{r=0}^{r=1} = \frac{12}{5}\pi .$$

Spherical Coordinates:

$0 \le \rho$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

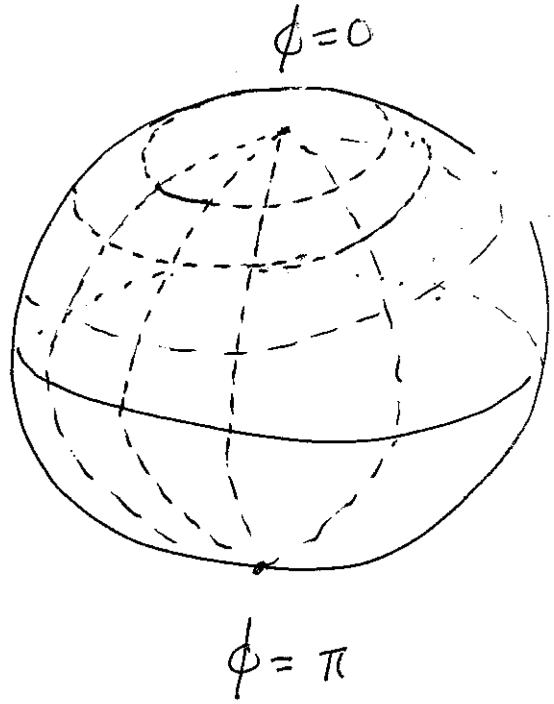


$x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$

Q: What is dV here?

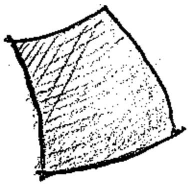
First, (θ, ϕ) give coordinates on the shere of radius ρ.

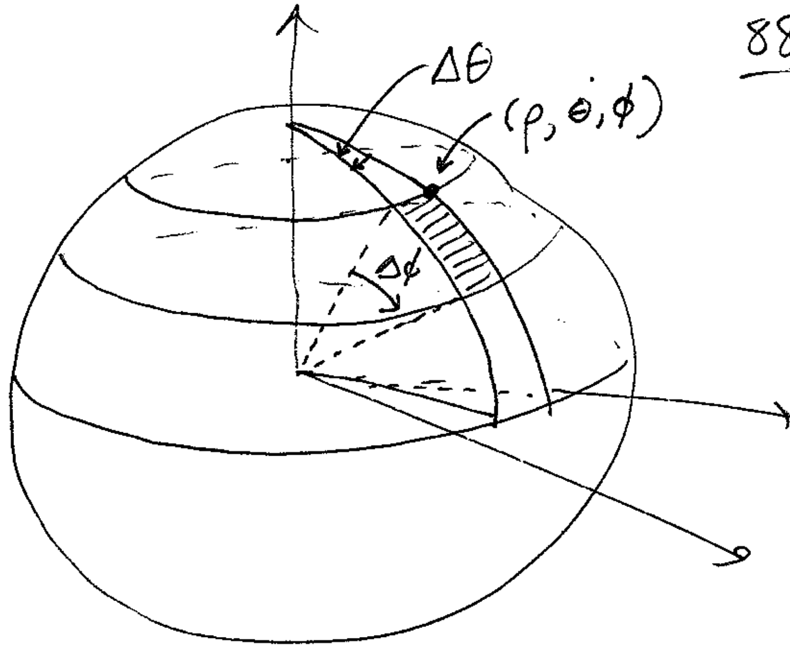
(like latitude/longitude)

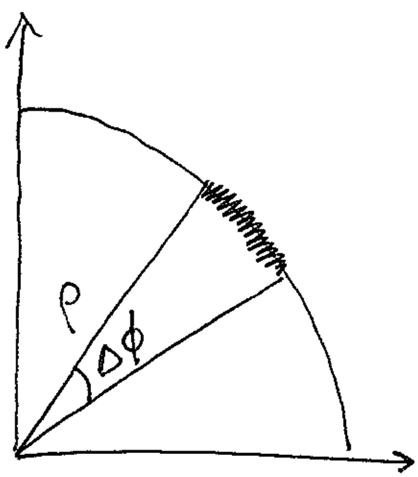


Q1: How much area is the region where θ changes by $\Delta\theta$ and ϕ changes by $\Delta\phi$?

How long are the four sides:

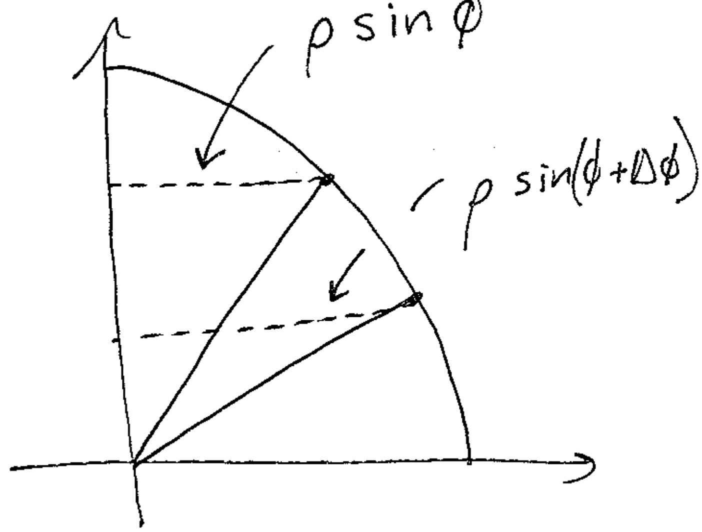






vertical sides $= \rho\,\Delta\phi$

Horizontal sides



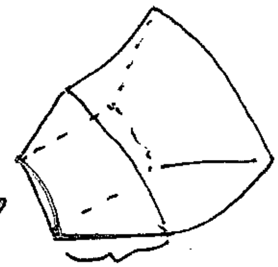
Top $= \Delta\theta\,\rho\sin\phi$

Bottom $= \Delta\theta\,\rho\sin(\phi+\Delta\phi)$

So, the approximate area is $\rho^2\,\Delta\theta\,\Delta\phi\,\sin\phi$

(Point: Basically a square, $\sin(\phi+\Delta\phi) \approx \sin\phi + \sin'(\phi)\,\Delta\phi + \dots$)

So spherical box with sides $\Delta\theta, \Delta\phi, \Delta\rho$



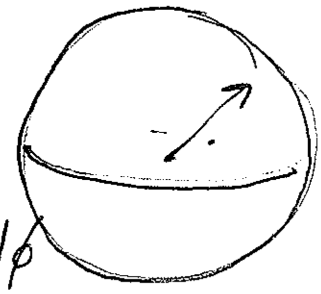
$\rho\Delta\phi$ $\Delta\rho$

has volume.

$$\rho^2 \sin\phi \, \Delta\rho \Delta\theta \Delta\phi.$$

Conclusion: $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi.$

Ex: Volume of sphere of radius 1



$$\iiint_R 1 \, dV = \int_0^{\pi}\int_0^{2\pi}\int_0^1 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{\rho^3}{3}\sin\phi \Big|_{\rho=0}^{1} d\theta \, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{1}{3}\sin\phi \, d\theta \, d\phi = \frac{2}{3}\pi \int_0^{\pi} \sin\phi \, d\phi$$

$$= \frac{2}{3}\pi \left(-\cos\phi\right)\Big|_{\phi=0}^{\pi} = \frac{2}{3}\pi\left(--1-(-1)\right)$$

$$= \frac{4\pi}{3}.$$ Same as last time!