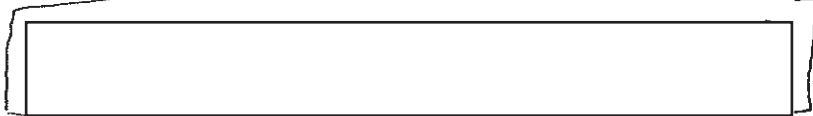


Lecture 22: Conservative Vector Fields (16.3)

Previously on Math 241:



Fund. Thm. of Line Integrals: $f: \mathbb{R}^n \rightarrow \mathbb{R}$ differentiable

C a curve joining A to B .

$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

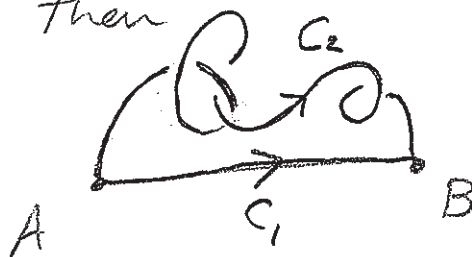


A vector field $\vec{F}: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is conservative if $\vec{F} = \nabla f$
for some $f: \mathbb{R}^n \rightarrow \mathbb{R}$

By the Fund Thm, if \vec{F} is conservative then

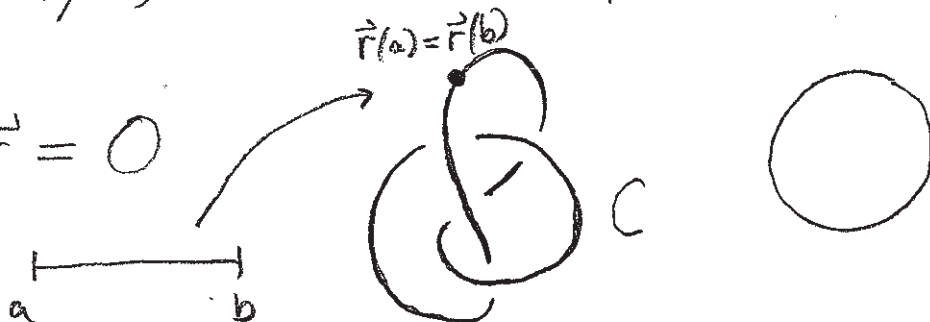
(a) Independence of path: If C_1 and C_2
are two paths joining A to B , then

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$



(b) If C is a closed curve (i.e. starts and ends
at the same pt, like a circle) then

$$\int_C \vec{F} \cdot d\vec{r} = 0$$



Reason: If $\vec{r}(a) = \vec{r}(b)$, then

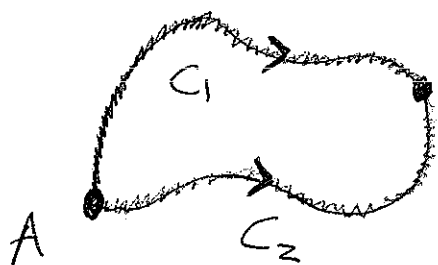
Skip ahead to the examples.

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = 0$$

↑ same input

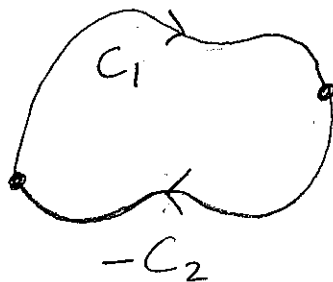
Note: Actually these two conditions are equivalent.

For example if (b) is true for \vec{F} consider



Let $-C_2$ denote C_2 backwards

Take $C =$



Then

$$0 = \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r}$$

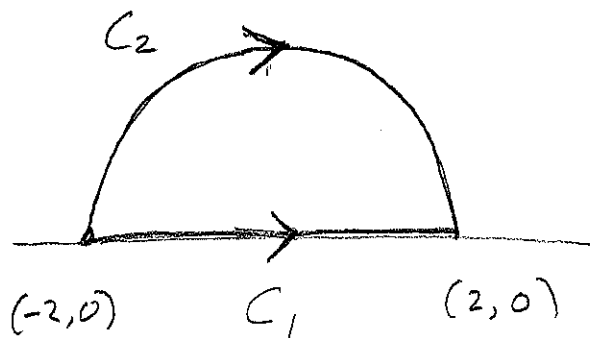
$$\Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

[Conversely, if (a) holds for \vec{F} , then break a closed curve into 2 segments to see that $\int_C \vec{F} \cdot d\vec{s} = 0$]

Examples of nonconservative vector fields:

68.

① $\vec{F} = (y, 0)$ [from yesterday's worksheet]

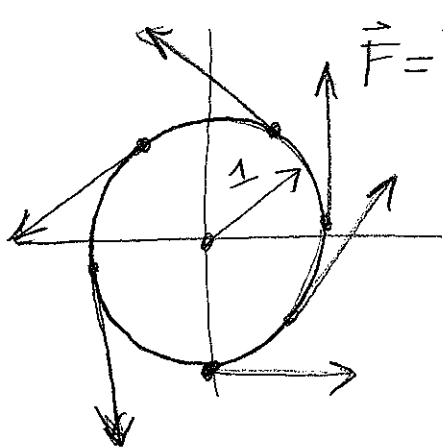


$$\int_{C_1} \vec{F} \cdot d\vec{r} = 0$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = 2\pi$$

} different!

② $\vec{F} = (-y, x)$ [from 1st lecture on vector fields.]



$$\vec{F} = \vec{T} \quad \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T} ds$$

$$= \int_C ds = 2\pi \neq 0$$

Note: Being conservative is kinda subtle. For example, is $\vec{F} = (y, x)$ conservative? [Compare ② above]

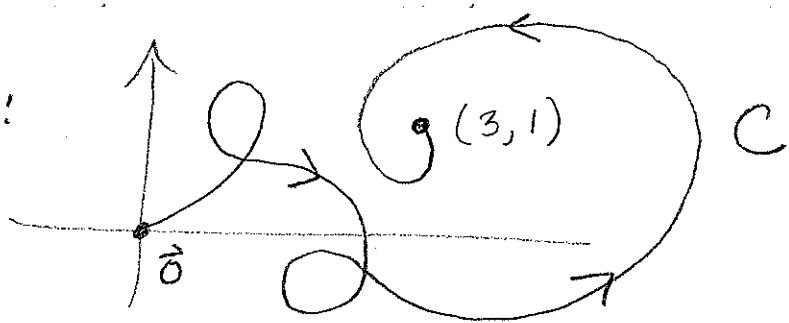
A. Yes. Need $f(x, y)$ with $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = x$

First cond says $f = \int y dx = xy + C(y)$

↑ treated as a constant

Second is similar, so take $f = xy$.

Usefulness is clear:



$$\int_C \vec{F} \cdot ds = f(3,1) - f(0,0) = 3.$$

[Will give two tests for \vec{F} to be conservative, but first need to introduce some terms...]

Focus on a vector field \vec{F} on \mathbb{R}^2 , with domain D .

Ex: $\vec{F} = (-y, x)$ $D = \mathbb{R}^2$

$$\vec{F} = \frac{1}{x^2+y^2}(-y, x) \quad D = \{(x,y) \neq (0,0)\}$$

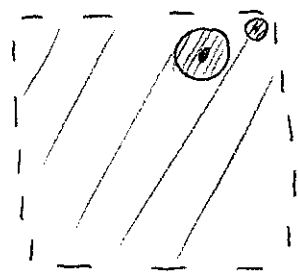
Properties D may have:

Open: Roughly, D contains none of its boundary points. Precisely, each pt of D is the center of a disc also contained in D .

Ex: $D = \{(x,y) \mid 0 < x < 1, 0 < y < 1\}$

$D = \mathbb{R}^2$

$D = \{(x,y) \neq (0,0)\}$

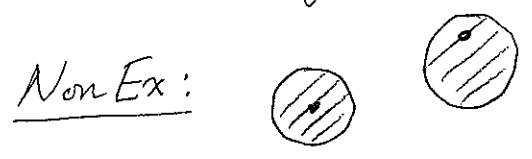
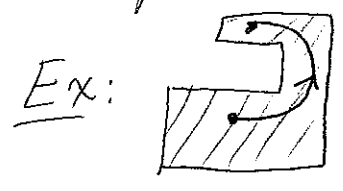


Closed: Opposite of open.

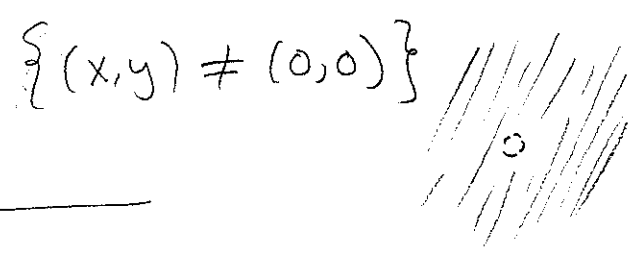
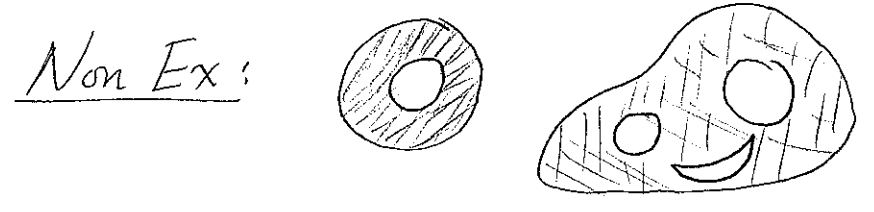
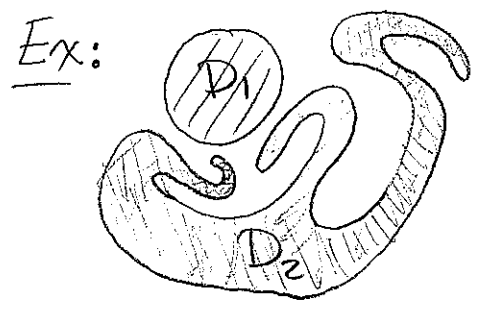
[D is closed, if and only if its complement is open.]

Connected: Any two points in D can be joined

by a path inside D.



Simply Connected: Connected + no holes.



Thm A: Suppose \vec{F} is a vector field on an open connected set D. Then \vec{F} is conservative if and only if $\int_C \vec{F} \cdot d\vec{r}$ is path independent.

Thm B: Suppose $\vec{F} = (P, Q)$ is a vector field on an open simply connected set D. Then \vec{F} is conservative if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D.

[Thm A is nice, but hard to apply. Will talk about next time.]

Reason for Thm B: Suppose $\vec{F} = \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$

Then

" P Q

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$$

and

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} \quad \text{and mixed partials are equal!}$$

[We'll come back to why $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \vec{F}$ conserv. just before Thanksgiving...]

Ex: $\vec{F} = (y, x)$ and $\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x} \Rightarrow$ conserv.

$\vec{F} = (-y, x)$ and $\frac{\partial P}{\partial y} = -1 \neq 1 = \frac{\partial Q}{\partial x} \Rightarrow$ not conserv.

Ex: $\vec{F} = \frac{1}{x^2+y^2}(-y, x)$ on $D = \{(x, y) \neq 0\}$

On HW: Check that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ but not path indep, hence

Point: D is not simply conn.] not conservative.