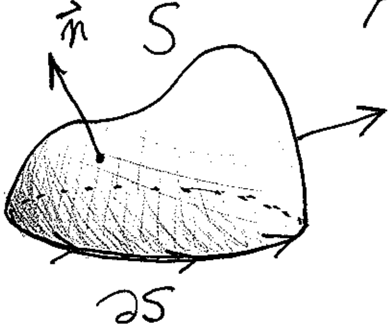
Lecture 40: More on Stokes' Theorem

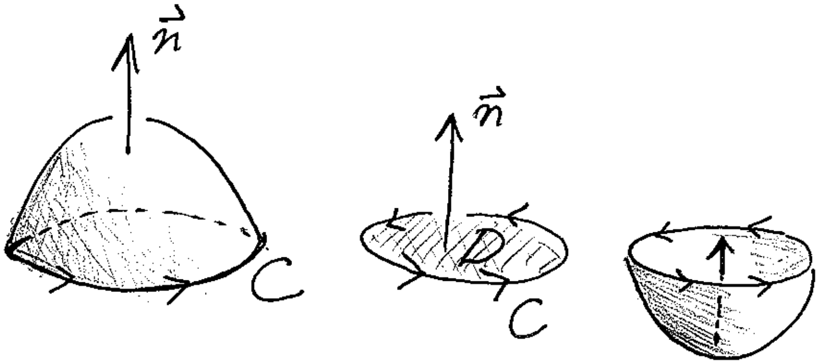
<u>Last time</u>: <u>Stokes' Thm</u>: S surface in $\mathbb{R}^3$

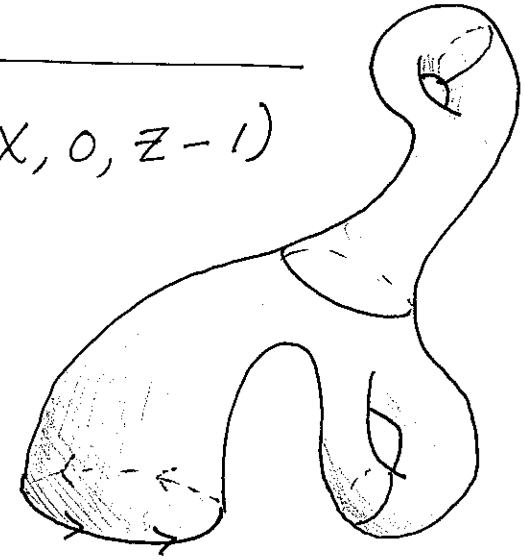
$\vec{F}:\mathbb{R}^3 \to \mathbb{R}^3$ vector field. Then



$$\int_{\partial S} \vec{F}\cdot d\vec{r} = \iint_S (\operatorname{curl}\vec{F})\cdot\vec{n}\, dA$$

<u>Ex</u>: $\vec{F} = (y, xz, 1)$ $\quad \operatorname{curl}\vec{F} = (-x, 0, z-1)$





$$\iint_S (\operatorname{curl}\vec{F})\cdot\vec{n}\, dA = -\pi = \int_C \vec{F}\cdot d\vec{r} \quad \text{for } \underline{\text{all}} \text{ of these.}$$

[Takes some getting used to, is really just Green's Thm / Divergence Thm in disguise...]

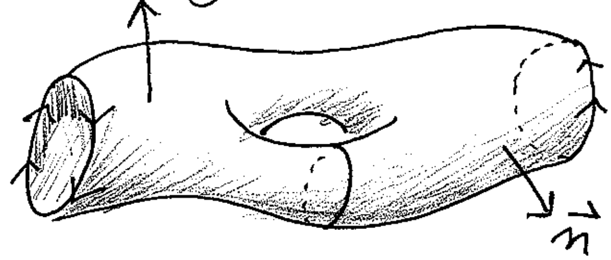
<u>Check the easy one</u>:

since $z=0$

$$\iint_D (\operatorname{curl}\vec{F})\cdot\vec{n}\, dA = \iint_D (-x, 0, -1)\cdot(0,0,1)\, dA$$

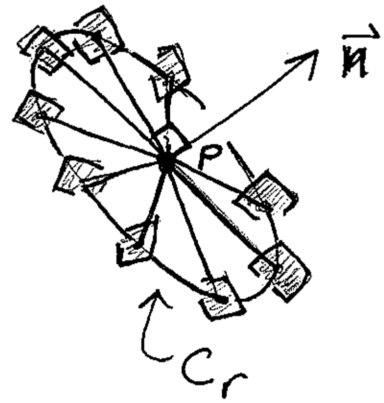
$$= \iint_D -1\, dA = -\text{Area}(D) = \boxed{-\pi} \checkmark$$

Note: Also works when S has several boundary components, [provided they are oriented correctly.]



Understanding Curl: Consider a small paddle wheel at P, of radius r.

$\vec{F}$ = fluid flow

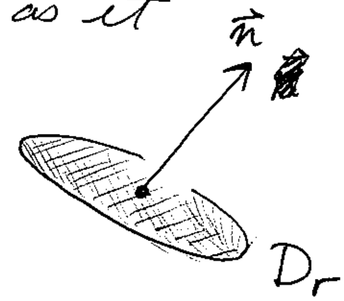


Q: How fast does it spin?

A: $$\omega = \frac{1}{2\pi r^2} \int_{C_r} \vec{F} \cdot d\vec{r}$$

$$\boxed{\text{units} = \frac{\text{radians}}{\text{time}}}$$

Plausible: (a) Want tangential component of $\vec{F}$ as it hits the paddles.
(b) Looks like an average (almost).



Stokes says:

$$\omega = \frac{1}{2\pi r^2} \iint_{D_r} (\text{curl}\, \vec{F}) \cdot \vec{n} \, dA$$

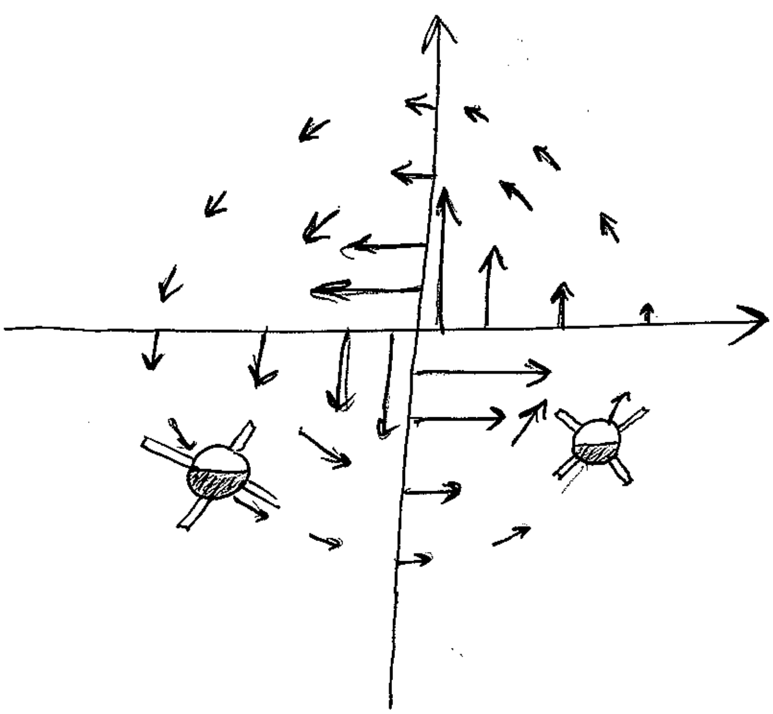
$$= \frac{1}{2}\left(\frac{1}{\text{Area}(D_r)} \iint_{D_r} (\text{curl } \vec{F}) \cdot \vec{n} \, dA \right)$$

Taking $r \to 0$ get: $\omega = \frac{1}{2}(\text{curl } \vec{F}) \cdot \vec{n}$

So the rate of rotation is fastest the direction of $\text{curl } \vec{F}(p)$ and then $\omega = \frac{1}{2}|\text{curl } \vec{F}|$.

<u>Note</u>: A vector field where $\text{curl } \vec{F} = \vec{0}$ everywhere are called <u>irrotational</u>.

<u>Ex</u>: $\vec{F} = \frac{1}{x^2+y^2}(-y, x, 0)$ has $\text{curl } \vec{F} = \vec{0}$ except at $(0,0)$ where it's not defined.



Experimentally, a draining tub is an irrotational flow!

Conservative Vector Fields: $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$

is conservative if $\vec{F} = \nabla f$ for some $f: \mathbb{R}^n \to \mathbb{R}$

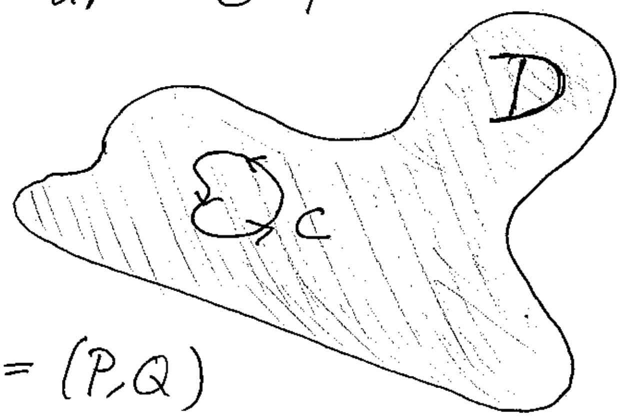
Ex: $\vec{F} = (x, y)$ is conservative since

$$= \nabla\left(\tfrac{1}{2}(x^2+y^2)\right)$$

$\vec{F} = (-y, x)$ is not conservative since

P, Q $$\frac{\partial Q}{\partial x} = 1 \neq -1 = \frac{\partial P}{\partial y}$$

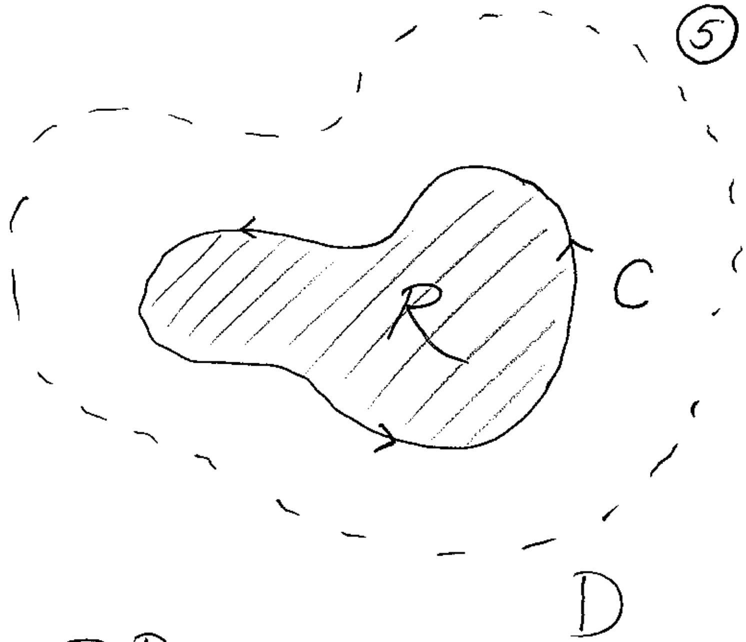
Thm A: $\vec{F}$ on a connected set D in $\mathbb{R}^n$ is conservative if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve C.



in $\mathbb{R}^2$

Thm B: If D is simply connected (no holes) then $\vec{F} = (P, Q)$ is conservative if and only if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$.

Missing Link: If $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ then $\int_C \vec{F} \cdot d\vec{r} = 0$ for each closed curve C. Hence $\vec{F}$ is conservative by Theorem A.



Reason: As D has no holes, the curve C is the boundary of a region R [where $\vec{F}$ makes sense.] Then

$$\int_C \vec{F}\cdot d\vec{r} = \iint_R \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{=0!} \, dA = 0.$$

Next time: What is Thm B for $\mathbb{R}^3$?

Suppose $\vec{F} = \nabla f = (f_x, f_y, f_z)$

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \partial f/\partial z \end{vmatrix} = \Big(\underbrace{\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}}_{0},\ 0,\ 0\Big)$$

Q: Is this enough?