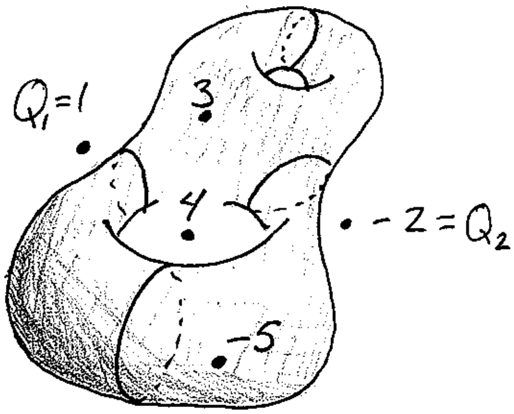
Lecture 43: Maxwell's Equations

Last time: Charges Q_i at positions $\vec{p}_i$

$$\vec{E}(\vec{r}) = \sum \frac{Q_i}{4\pi\varepsilon_0} \frac{1}{|\vec{r}-\vec{p}_i|^3}(\vec{r}-\vec{p}_i)$$



Gauss's Law: D a region in $\mathbb{R}^3$

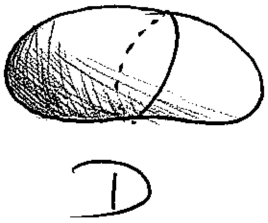
$$\iint_{\partial D} (\vec{E}\cdot\vec{n})\, dA = \frac{1}{\varepsilon_0}\left(\text{Total charge } \underline{\text{in}}\ D\right)$$

$$\text{Flux} = \frac{-2}{\varepsilon_0}$$

* * *

[On HW for today, used this compute the total charge from a formula for $\vec{E}$. This is not impractical... When there are many (e.g. 10^{20}) can't use ~~the~~ the formula for $\vec{E}$ directly. Don't want to focus on each charge individually just as we don't count molecules when] measuring the mass.

Mass: $\rho(x,y,z)$ mass density, units $= \frac{g}{m^3}$

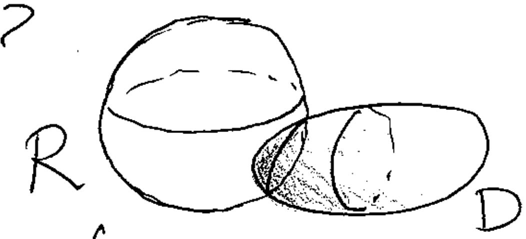


$$\text{Total Mass of } D = \iiint_D \rho\, dV$$

Charge: $\rho(x,y,z)$ charge density, units $= \frac{\text{Coulomb}}{m^3}$

$$\text{Total Charge of } D = \iiint_D \rho\, dV$$

<u>Q</u>: How does ρ determine $\vec{E}$?



Gauss's Law should still hold, so for a region R we have

$$\frac{1}{\varepsilon_0}\begin{pmatrix}\text{Charge}\\ \text{in } R\end{pmatrix} = \iint_{\partial R} (\vec{E}\cdot\vec{n})\, dA \underset{\nwarrow \text{ Divergence Thm}}{=} \iiint_R \operatorname{div} \vec{E}\, dV$$

$$\shortparallel$$

$$\frac{1}{\varepsilon_0}\iiint_R \rho\, dV$$

As true for <u>all</u> regions R, must have $\operatorname{div}\vec{E} = \frac{\rho}{\varepsilon_0}$

[<u>Q</u>: Does this answer the question? Not completely since many vector fields have the <u>same</u> divergence.]

<u>A</u>. $\vec{E}(\vec{r}) = (E_1(\vec{r}), E_2(\vec{r}), E_3(\vec{r}))$ where if $\vec{r} = (a,b,c)$

then

$$E_1(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_D \frac{(a-x)\,\rho(x,y,z)}{|\vec{r}-(x,y,z)|^3}\, dV$$

etc.

<u>Exercise</u>: Take D =  unit sphere and $\rho = 1$.

Use above to calculate the electric field at $\vec{r} = (a,b,c)$

<u>Hint</u>: Use symmetry to reduce to the case where $\vec{r} = (a,0,0)$.

Maxwell's Equatrons:

$\vec{E}(x,y,z,t)$ - Electric field (at time t) ($\mathbb{R}^4 \to \mathbb{R}^3$)

$\vec{B}(x,y,z,t)$ - Magnectic field

$\rho(x,y,z,t)$ - charge dnsity ($\mathbb{R}^4 \to \mathbb{R}$)

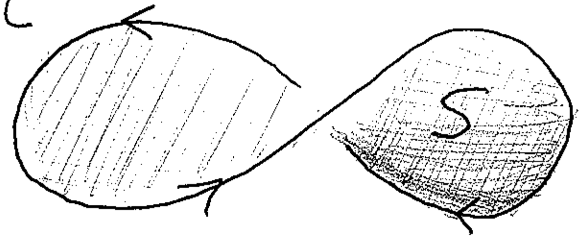
Gauss's Law:

$$\operatorname{div} \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \iint_{\partial R} \vec{E}\cdot\vec{n}\, dA = \iiint_R \rho\, dV$$

Gauss's Law for magnetic fields: [No magnectic monopoles.]

$$\operatorname{div} \vec{B} = 0 \qquad \iint_{\partial R} (\vec{B}\cdot\vec{n})\, dA = 0$$

Faraday's Law of Induction: A changing magnetic field induces a current in a loop of wire.



$$\underbrace{\int_C \vec{E}\cdot d\vec{r}}_{\text{electromotive force aka voltage}} = -\frac{\partial}{\partial t}\iint_S \vec{B}\cdot\vec{n}\, dA$$

Now

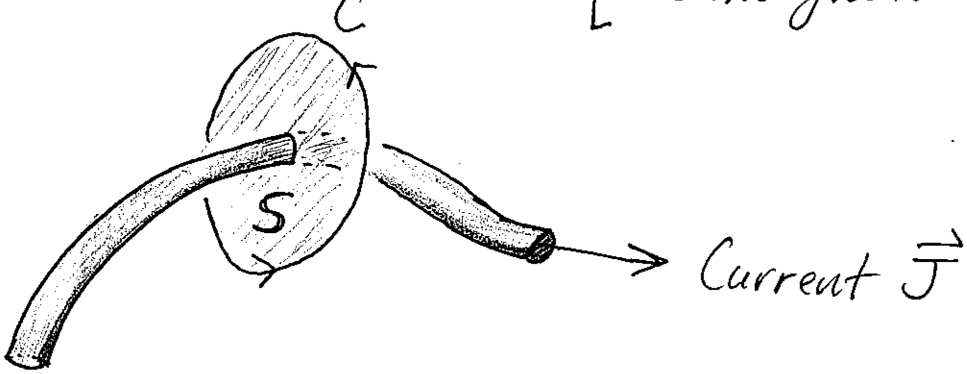
$$\int_C \vec{E}\cdot d\vec{r} = \iint_S (\operatorname{curl}\vec{E})\cdot\vec{n}\, dA \quad \text{and so} \quad \boxed{\operatorname{curl}\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

Unit check: ⓐ $\vec F = Q\vec E \Rightarrow \vec E$ in $\frac{N}{C} = \frac{V}{m} \Rightarrow \int_C \vec E\cdot \underbrace{d\vec r}_{m}$ is in volts.

ⓑ $\vec B$ has units T = Tesla = $\frac{Vs}{m^2}$ $\quad\vec n$ = unitless

So $\iint_S \vec B\cdot\vec n \underbrace{dA}_{m^2}$ is in Vs $\Rightarrow \frac{\partial}{\partial t}\iint_S(\vec B\cdot\vec n)\,dA$ is in V.

Ampere's circuital law: [Current in a wire or a changing electric field induces a magnetic field.]



$$\int_C \vec B\cdot d\vec r = \mu_0\iint_S(\vec J\cdot\vec n)\,dA + \varepsilon_0\mu_0\frac{\partial}{\partial t}\iint_S(\vec E\cdot\vec n)\,dA$$

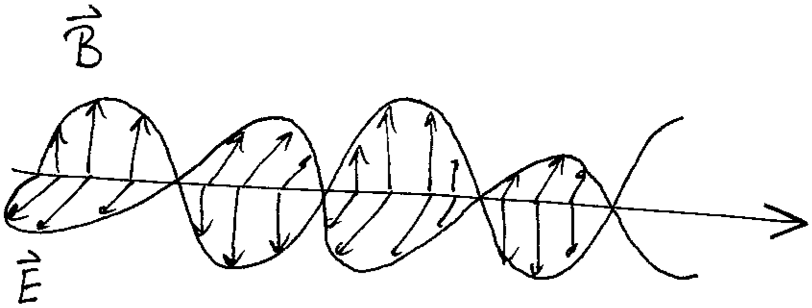
$$\|$$

$$\iint_S(\text{curl}\,\vec B)\cdot\vec n\;dA$$

So $\boxed{\text{curl}\,\vec B = \mu_0\vec J + \varepsilon_0\mu_0\frac{\partial\vec E}{\partial t}}$

ε_0 = permittivity of free space = farads/m $\qquad \mu_0$ = permeability of free space = N/A^2

And there was light...



C = speed of light

$= \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$

$C = \omega \lambda$ ← wave length

↑ freq

$$\vec{E} = (0,\ C \cdot \cos(2\pi(\omega t - x/\lambda)),\ 0)$$

$$\vec{B} = (0,\ 0,\ \cos(2\pi(\omega t - x/\lambda)))$$

Check: $\text{curl}\,\vec{E} = \frac{\partial \vec{B}}{\partial t}$ $\quad$ $\text{curl}\,\vec{B} = \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$

$\text{div}\,\vec{E} = 0$ $\quad$ $\text{div}\,\vec{B} = 0.$

[Moral: Here are other applications of line and surface integrals, as well as the relations between them.]

If time remains: ⓐ Multitude of integral theorems.

ⓑ $\int_{\partial M} \omega = \int_M d\omega$