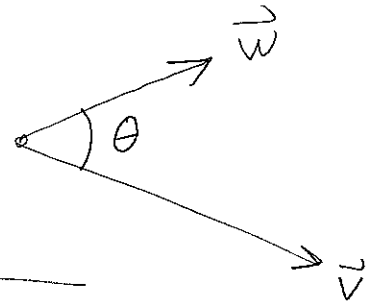


Lecture 4: Cross Product (Section 12.4)

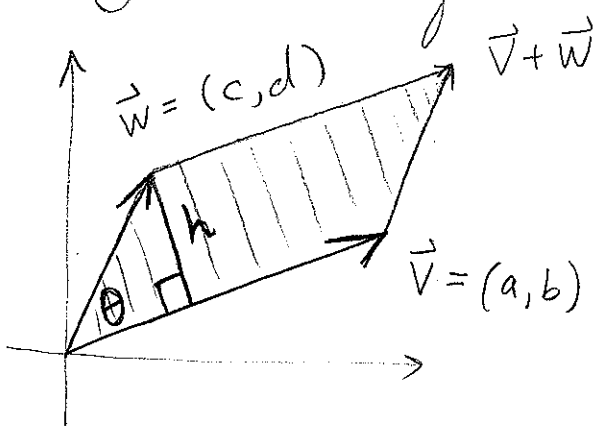
Last time: $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3)$

$$\begin{aligned} \vec{v} \cdot \vec{w} &= v_1 w_1 + v_2 w_2 + v_3 w_3 \\ &= |\vec{v}| |\vec{w}| \cos \theta \end{aligned}$$



Determinant: $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Meaning: Area of

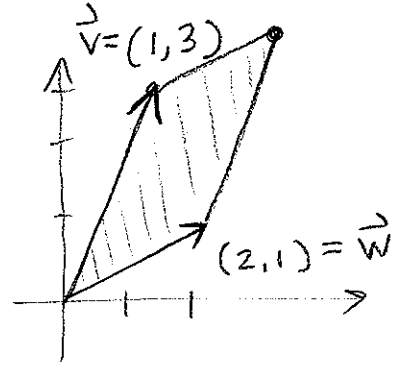


$$\begin{aligned} A &= (\text{base})(\text{height}) \\ &= |\vec{v}| |\vec{w}| \sin \theta \end{aligned}$$

Reason:

$$\begin{aligned} A^2 &= |\vec{v}|^2 |\vec{w}|^2 \sin^2 \theta = |\vec{v}|^2 |\vec{w}|^2 (1 - \cos^2 \theta) \\ &= |\vec{v}|^2 |\vec{w}|^2 - (\vec{v} \cdot \vec{w})^2 \\ &= (a^2 + b^2)(c^2 + d^2) - (ac + bd)^2 \\ &= a^2 d^2 + b^2 c^2 - 2abcd = (ad - bc)^2 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}^2 \end{aligned}$$

Ex:



$$\text{Area} = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 2 \cdot 3 = -5$$

negative area??

Switch \vec{v} and \vec{w} : $\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 2 \cdot 3 - 1 = 5$

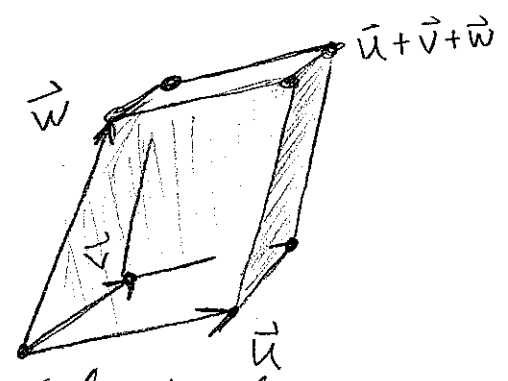
In general

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{cases} \text{area} & \text{if } \vec{w} \text{ is anticlockwise from } \vec{v} \\ -\text{area} & \text{if } \vec{w} \text{ is clockwise from } \vec{v}. \end{cases}$$

[First example of the right-hand rule.]

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

= Volume of
in \mathbb{R}^3
of this parallelepiped.



Cross product: $\vec{V} = (v_1, v_2, v_3) = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}$

(14)

$$\vec{W} = (w_1, w_2, w_3)$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} \vec{i} - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} \vec{j} + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \vec{k}$$

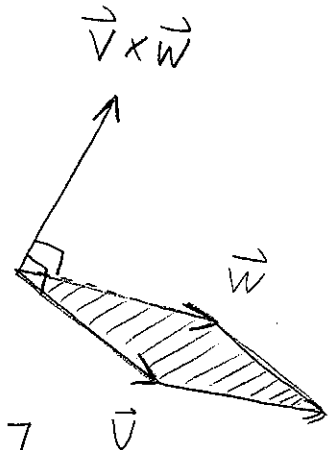
Ex: $\vec{V} = (0, 1, 3)$ $\vec{W} = (2, 1, 1)$

$$\vec{V} \times \vec{W} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} \vec{k}$$
$$= -2\vec{i} + 6\vec{j} - 2\vec{k} = (-2, 6, -2)$$

Usefulness: Maxwell's Equations, torque, geom. prob.

Properties:

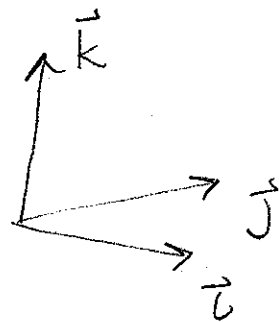
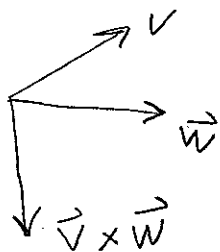
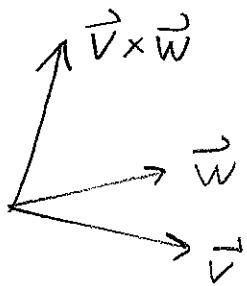
- 1) $\vec{V} \times \vec{W}$ is orthogonal to \vec{V} and \vec{W}
- 2) $|\vec{V} \times \vec{W}| = |\vec{V}| |\vec{W}| \sin \theta$
= area of parallelogram
- 3) $\vec{V} \times \vec{W} = -\vec{W} \times \vec{V}$ [Not commutative!]



4) $\vec{V} \times \vec{V} = \vec{0}$

5) $\vec{u} \times (\vec{V} + \vec{W}) = \vec{u} \times \vec{V} + \vec{u} \times \vec{W}$

Note: ① and ② almost determine $\vec{v} \times \vec{w}$
 the rest comes from the right-hand rule



Ex: $\vec{i} \times \vec{j} = \vec{k}$ $\vec{j} \times \vec{k} = \vec{i}$
 $\vec{i} \times \vec{k} = -\vec{j}$

Not associative: $\vec{i} \times (\vec{i} \times \vec{j}) = \vec{i} \times \vec{k} = -\vec{j}$
 $(\vec{i} \times \vec{i}) \times \vec{j} = \vec{0} \times \vec{j} = \vec{0}$

Ex: Find the equation of the plane

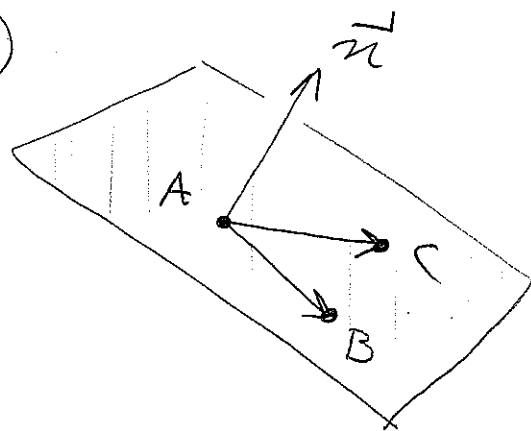
Containing $A = (0, 0, 1)$ $B = (0, 1, 4)$
 and $C = (2, 1, 2)$

$$\vec{v} = \vec{AB} = (0, 1, 3)$$

$$\vec{w} = \vec{AC} = (2, 1, 1)$$

Normal vector

$$\vec{n} = \vec{v} \times \vec{w} = (-2, 6, -2)$$



Equation for plane

$$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \quad \text{with} \quad \begin{aligned} \vec{r} &= (x, y, z) \\ \vec{r}_0 &= (0, 0, 1) \end{aligned}$$

So get

$$-2x + 6y - 2z + 2 = 0$$

Triple Product:

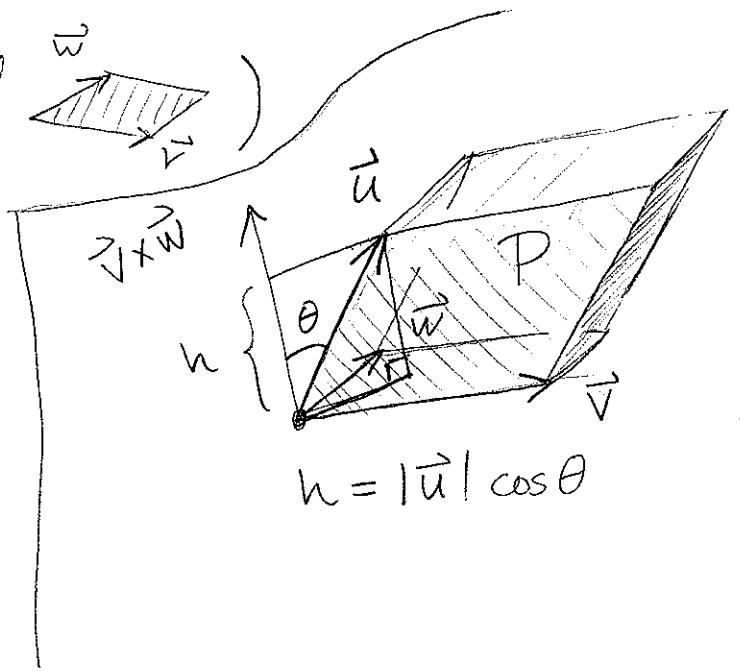
$$\vec{u} \cdot (\vec{v} \times \vec{w}) = |\vec{u}| |\vec{v} \times \vec{w}| \cos \theta$$

$$= (|\vec{u}| \cos \theta) (\text{Area of } \vec{v} \times \vec{w})$$

$$= (\text{height of } P) (\text{Area of Base of } P)$$

$$= (\text{Volume of } P)$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$



Check: Expand out $\vec{u} \cdot (\vec{v} \times \vec{w})$ and $\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

Reasons for props of cross product (only if asked)

① Compute $\vec{v} \cdot (\vec{v} \times \vec{w})$ and $\vec{w} \cdot (\vec{v} \times \vec{w})$
and note you get 0.

③-⑤ calculations.

② Basically the same calc as $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \text{Area}$,
see text for details.

_____ o _____

Q: Is there a better way to mult vectors in \mathbb{R}^3 ?

Would like it to be associative, $\vec{v} \star \vec{w} \neq \vec{0}$ if $\vec{v}, \vec{w} \neq \vec{0}$.

A. No. All there is: $\mathbb{R} = \mathbb{R}^1$, $\mathbb{C} = \mathbb{R}^2$
 $\mathbb{H} = \mathbb{R}^4$, $\mathbb{O} = \mathbb{R}^8$

Connected to the fact that if you
comb a hairy sphere,
there is always a
whirl...

