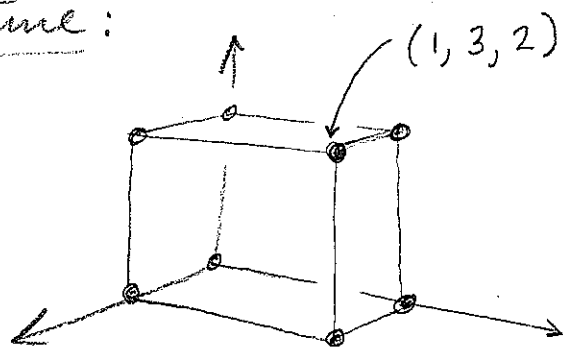


Lecture 2: Vectors (Section 12.2)

(4)

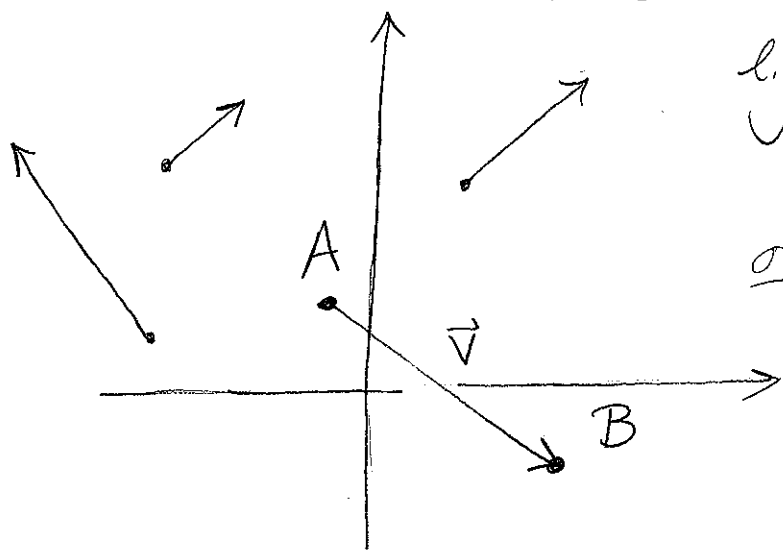
Last time:

\mathbb{R}^3 :



$\mathbb{R}^n =$ tuples
 (x_1, x_2, \dots, x_n)
of real #s.

Vectors in \mathbb{R}^2 : arrow where both direction and length are important



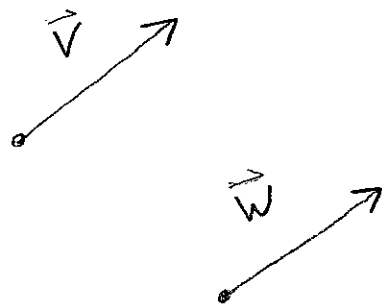
e.g. use to denote
windspeed and direction
or record relative
positions of two
points.

Denoted \vec{v} or \vec{w} or similar. [bold used in books]

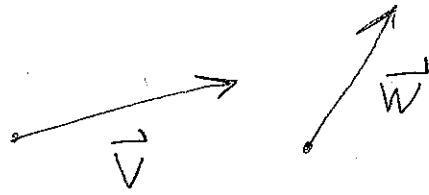
\vec{v} and \vec{w} are regarded as equal if they are the same up

to translation.

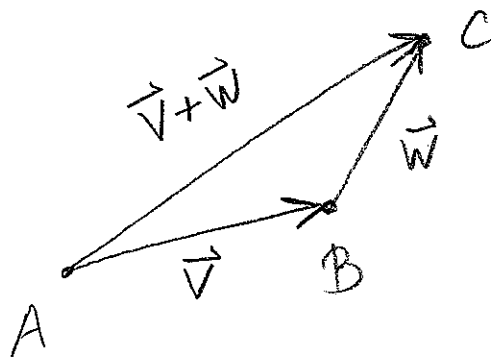
$$\vec{v} = \vec{w}$$



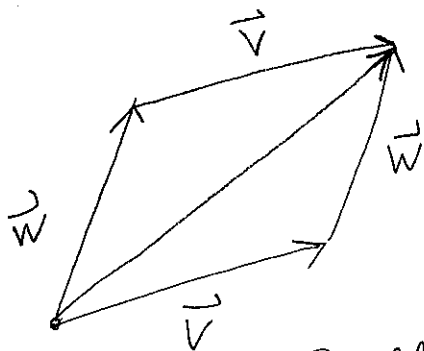
Addition:



Start at A, head along \vec{v} to B, then along \vec{w} to C.



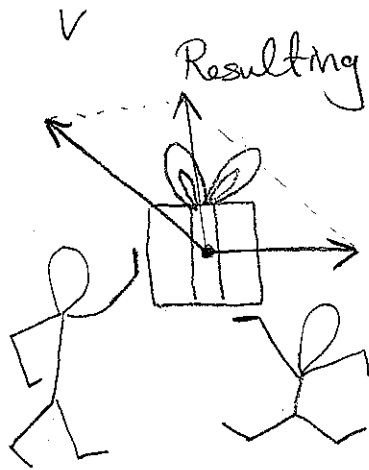
Note



and so

$$\vec{v} + \vec{w} = \vec{w} + \vec{v}$$

Force:

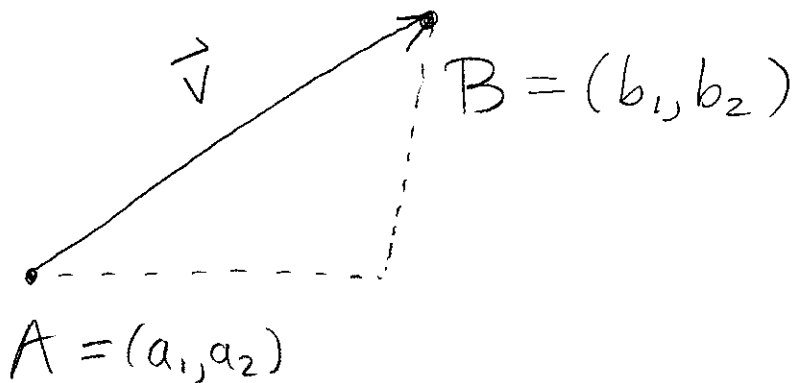


Resulting force is the sum of the two forces.

Like points in \mathbb{R}^2 , vectors are param.

by pairs of #s.

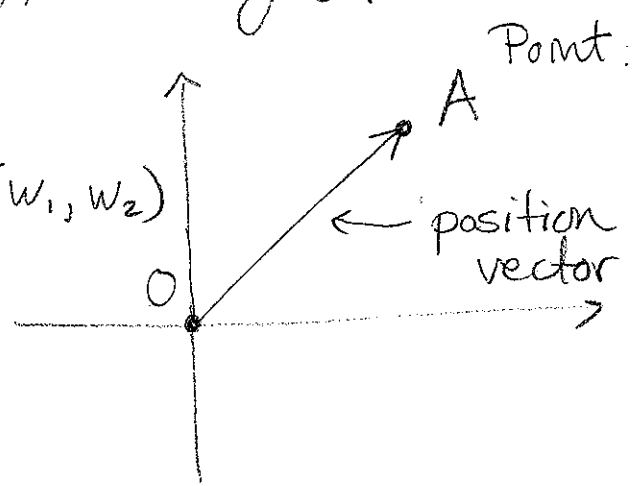
$$\vec{v} = (b_1 - a_1, b_2 - a_2)$$



Points and vectors are diff. though.

def $\vec{v} = (v_1, v_2)$ and $\vec{w} = (w_1, w_2)$

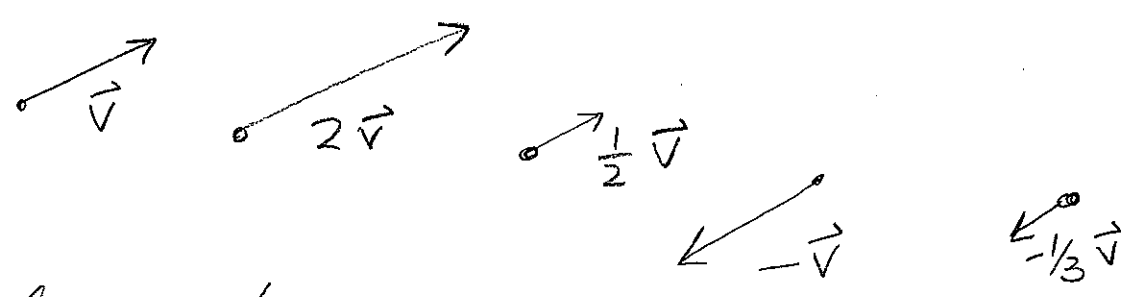
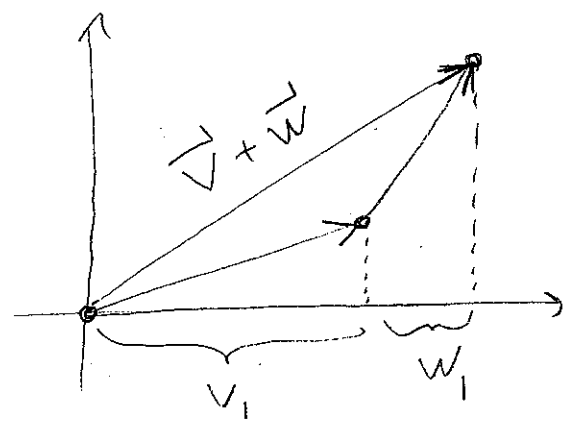
then



$$\vec{v} + \vec{w} = (v_1 + w_1, v_2 + w_2)$$

Scalar multiplication:

$c \vec{v}$ = same direction
 ↑ vector length scaled by c .

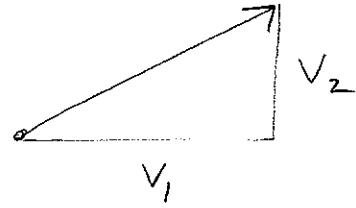


in coordinates

$$c \vec{v} = (c v_1, c v_2) \text{ where } \vec{v} = (v_1, v_2)$$

Length:

$$|\vec{v}| = \sqrt{v_1^2 + v_2^2}$$



[Some books use $\|\vec{v}\|$.]

Properties: [Can work out from geom or alg def.]

$$\vec{v} + \vec{w} = \vec{w} + \vec{v} \quad (\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$$

$$\vec{v} + \vec{0} = \vec{v} \text{ if } \vec{0} = (0,0) \quad \vec{v} + (-1)\vec{v} = \vec{0}$$

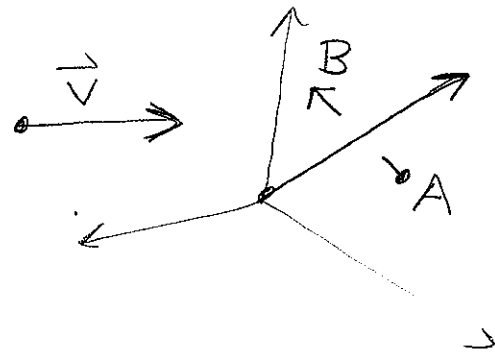
$$c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w} \quad (c+d)\vec{v} = c\vec{v} + d\vec{v}$$

$$(cd)\vec{v} = c(d\vec{v}) \quad 1\vec{v} = \vec{v}$$

Vectors in \mathbb{R}^3 : Same idea

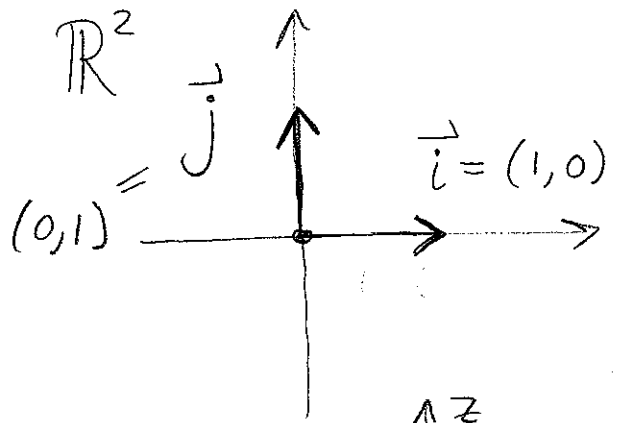
$$\vec{v} = (v_1, v_2, v_3)$$

Can add etc. as before.

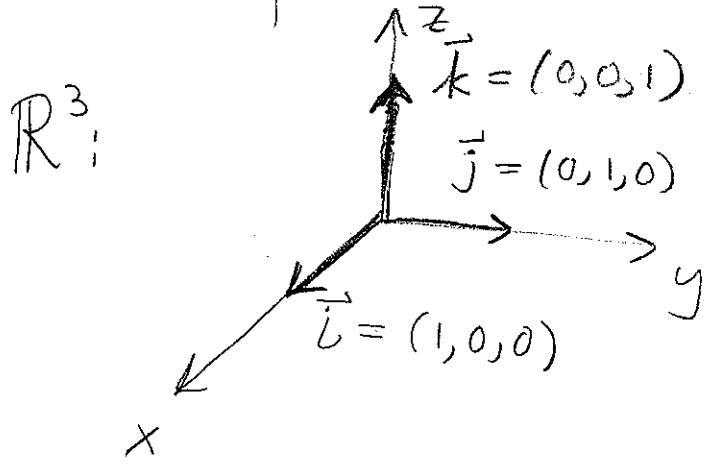


Vectors in \mathbb{R}^n : Once more, with feeling.

Standard vectors:



$$3\vec{i} + 2\vec{j} = (3, 0) + (0, 2) = (3, 2)$$



$$(1, 3, 2) = \vec{i} + 3\vec{j} + 2\vec{k}$$

Can we multiply vectors? In \mathbb{R}^3 there are two ways to do this, different from each other and also ordinary mult of #s.

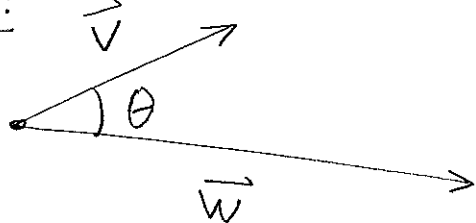
Dot Product: $\vec{V} = (v_1, v_2, v_3)$, $\vec{W} = (w_1, w_2, w_3)$
vectors in \mathbb{R}^3

$$\vec{V} \cdot \vec{W} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Ex: $\vec{v} = (1, 2, 0)$, $\vec{w} = (-1, 0, +2)$

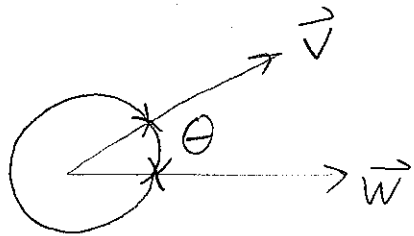
$$\vec{v} \cdot \vec{w} = 1 \cdot (-1) + 2 \cdot 0 + 0 \cdot 2 = -1$$

Key:



$$\vec{v} \cdot \vec{w} = |\vec{v}| |\vec{w}| \cos \theta$$

$0 \leq \theta \leq \pi$ is the smaller of two angles.



Ex: $\cos \theta = \frac{\vec{v} \cdot \vec{w}}{|\vec{v}| |\vec{w}|} = \frac{-1}{5} \Rightarrow \theta \approx 101.54^\circ$

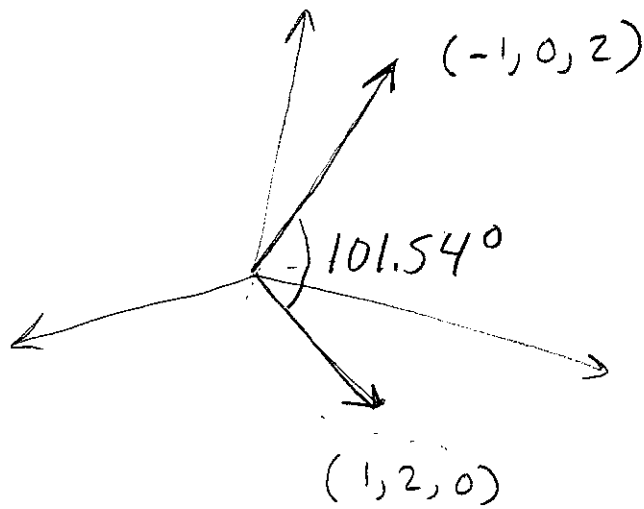
Note: Thus

$$\vec{v} \cdot \vec{w} = 0$$

exactly when $\cos \theta = 0$,

i.e. $\theta = \pm \pi/2$ and the

vectors meet at right angles.



Properties [Easy to see from the def.] (8)

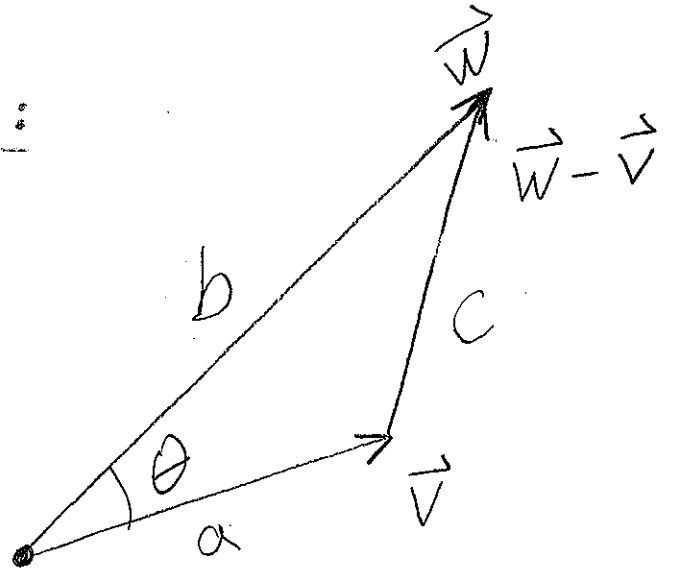
$$|\vec{v}|^2 = \vec{v} \cdot \vec{v} \quad \vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$$

$$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v} \quad (c \vec{v}) \cdot \vec{w} = c (\vec{v} \cdot \vec{w})$$

Idea behind key formula:

Law of cosines:

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$



So:

$$\begin{aligned} c^2 &= |\vec{w} - \vec{v}|^2 = (\vec{w} - \vec{v}) \cdot (\vec{w} - \vec{v}) \\ &= \vec{w} \cdot \vec{w} - \vec{w} \cdot \vec{v} - \vec{v} \cdot \vec{w} + \vec{v} \cdot \vec{v} \\ &= |\vec{w}|^2 + |\vec{v}|^2 - 2\vec{v} \cdot \vec{w} \\ &= a^2 + b^2 - 2\vec{v} \cdot \vec{w} \end{aligned}$$

$$\begin{aligned} \text{Thus } \vec{v} \cdot \vec{w} &= ab \cos \theta \\ &= |\vec{v}| |\vec{w}| \cos \theta. \end{aligned}$$