Lecture 11 : Multiplication as a linear transformation.

Last time: $F \subseteq K_1, K_2 \subseteq L$

Compositum: $K_1 K_2$ = smallest subfield of L containing K_1 and K_2

Thm: $[K_1K_2 : F] \leq [K_1 : F][K_2 : F]$

[In proving this thm, used an important idea which I'll now expand upon...]

Suppose R is an integral domain.
$F \subseteq R$ a subring which is also a field

Ex: F a field, $R = F[[t]] = \left\{ \sum_{n=0}^{\infty} a_n t \;\middle|\; a_n \in F \right\}$

↑ an int. domain (look at lowest degree terms.)

R isn't a field, e.g. t has no mult. inv.

Note: R is vector space over F, since

$f(r_1 + r_2) = fr_1 + fr_2$

$(f_1 + f_2)r = f_1 r + f_2 r$

$f_1(f_2 r) = (f_1 f_2) r$

$1_F r = r$

Uses that $1_F = 1_R$ since R is an integral domain.

Compare $R = \mathbb{R} \times \mathbb{R}$
$F = \mathbb{R} \times \{0\}$
$1_F \cdot (0, 2) = (0, 0)$.

Fix $r \in R$. Consider $T : R \longrightarrow R$

$$s \longmapsto rs$$

This is an F-linear transformation, since

$$T(fs) = rfs = f(rs) = f \cdot T(s)$$

$$T(s_1 + s_2) = r(s_1 + s_2) = rs_1 + rs_2 = T(s_1) + T(s_2)$$

Ex: $F = \mathbb{R}$, $R = \mathbb{C}$ $\quad r = a + bi$

$T_r : \mathbb{C} \to \mathbb{C}$ is given by the matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

in the $\mathbb{R}$-basis $\{1, i\}$ since

$$T_r(1) = a + bi, \quad T_r(i) = ai - b$$

Suppose $s \in \mathbb{C}$. Then

$$T_r(T_s(z)) = rsz = T_{rs}(z)$$

and

$$T_r(z) + T_s(z) = T_{r+s}(z)$$

This means that ring of 2×2 matrices with $\mathbb{R}$ coeffs.

$$\mathbb{C} \longrightarrow M_2(\mathbb{R})$$

$$r \longmapsto \text{Matrix of } T_r \text{ w.r.t to } \{1, i\}$$

That is $\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \;\middle|\; a,b \in \mathbb{R} \right\} \cong \mathbb{C}$

with usual ops of $+, \times$, inverses, etc.

In particular $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

"i"

<u>Thm</u>: R int domain containing a field F.
If $[R:F] = \dim_F R < \infty$, then R is a field.
(<u>Cor</u>: $\mathbb{C}$ is a field.)

<u>Proof</u>: Let $r \neq 0$ be in R. Consider $T: R \longrightarrow R$, $s \longrightarrow rs$

<u>Claim</u> T is onto.

If so $\exists\, s \in R$ with $T(s) = 1 \Rightarrow s = r^{-1}$!

Now T is onto $\iff \ker(T) = \{ s \in R \mid T(s) = 0 \}$
$= 0$.

Now if $s \in \ker(T)$, then $rs = 0 \Rightarrow s = 0$
as R is an int. domain. ▨

Now any inv. of a linear trans, gives an inv of $r \in R$. Not the matrix of $T_r : R \to R$, which depends on the basis but things like det, tr, char poly, ...

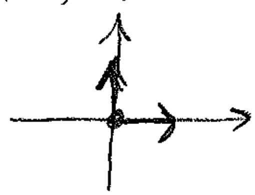
<u>Ex</u>: $F = \mathbb{R}$, $R = \mathbb{C}$

Note geom. meaning

$z = 2+3i \qquad T_z : \mathbb{C} \to \mathbb{C}$

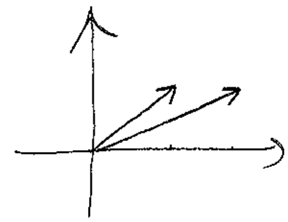
matrix w.r.t. $\{1, i\}$

$$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$$



matrix w.r.t. $\{1+i, 2+i\}$

$$\begin{pmatrix} 11 & 15 \\ -6 & -7 \end{pmatrix}$$



$$\det(T) = 2^2 + 3^2 = 13 = 11 \cdot (-7) + 6 \cdot 15.$$

In general, $z = a + bi$

$$\det(T_z) = \det\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a^2 + b^2 = |z|^2 = N(z).$$

<u>Ex</u>: $F = \mathbb{Q}$, $R = \mathbb{Q}(\sqrt{2})$

$\alpha = 3 + 2\sqrt{2}$ What is the minpoly of α?

Let $T = T_\alpha : R \to R$

$$T_\alpha(1) = \alpha \quad T_\alpha(\sqrt{2}) = 4 + 3\sqrt{2}$$

Matrix is $M = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$

Consider

$$p(x) = \text{char poly of } M$$
$$= \det(xI - M) = \det\begin{pmatrix} x-3 & -4 \\ -2 & x-3 \end{pmatrix}$$
$$= x^2 - 6x + 1.$$

Any matrix sat its char poly:

$$M^2 - 6M + I = 0$$

Also you can check that

$$p(\alpha) = 0$$

and hence p is the min poly of $\alpha / \mathbb{Q}$.

As before, the map $R \to M_2(\mathbb{Q})$

$$r \longmapsto \text{matrix of } T_r \text{ w.r.t. } \{1, \sqrt{2}\}$$

is 1-1 ring homomorphism.