Lecture 10:

Last time: K/F field extension, $\alpha \in K$ is algebraic over F if $\exists f(x) \in F[x]$ with $f(\alpha) = 0$.

K/F is algebraic if every elt of K is alg. over F.

Thm: $[K:F] < \infty \Rightarrow K/F$ is algebraic.

Thm: $\alpha, \beta \in K$ algebraic. Then $F(\alpha,\beta)/F$ is alg.

Cor: $\overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} \mid \alpha \text{ alg over } \mathbb{Q}\}$ is a field.

Thm: $F \subseteq K \subseteq L$ fields. Then $[L:F] = [L:K][K:F]$

Pf: If $[L:F] < \infty$, so is $[L:K]$ and $[K:F]$.

So assume $[L:K]$ and $[K:F]$ are finite.

$$\begin{array}{l} L \supset \{\beta_1, \dots, \beta_n\} \text{ a } K\text{-basis for } L. \\ | \\ K \supset \{\alpha_1, \dots, \alpha_m\} \text{ a } F\text{-basis for } K. \\ | \\ F \end{array}$$

Then $\{\gamma_{ij} = \alpha_i \beta_j\}$ are $n \cdot m$ elts which F-span L.

Claim: $\{\gamma_{ij}\}$ are linearly indep, and so form a basis.
Hence $[L:K] = nm$, as needed.

Suppose $a_{ij} \in F$ with $\sum_{i,j} a_{ij}\gamma_{ij} = 0$

Then $\sum_j \underbrace{\left(\sum_i a_{ij}\alpha_i\right)}_{\text{in } K}\beta_j = 0$ and so as the β_j are K-linearly indep,

must have each $\sum_i \underbrace{a_{ij}}_{\text{in } F}\alpha_j = 0$. As α_j are F-lin indep, get all $a_{ij} = 0$. So $\{\gamma_{ij}\}$ are F-lin. indep. $\blacksquare$

Finite Extension: K/F with $[K:F] < \infty$.

Cor: $F \subseteq K \subseteq L$. If L/K and K/F are finite so is L/F.

Thm $F \subseteq K \subseteq L$. If L/K and K/F are algebraic, so is L/F.

<u>Proof</u>: A given $\beta \in L$ is a root of some

$$p(x) = x^n + \alpha_{n-1}x^{n-1} + \cdots + \alpha_1 x + \alpha_0 \in K[x]$$

Consider $F \subseteq F(\alpha_0) \subseteq F(\alpha_1, \alpha_0) \subseteq \cdots$

$$\subseteq F(\alpha_0, \alpha_1, \ldots, \alpha_n) \subseteq F(\alpha_0, \alpha_1, \ldots, \alpha_n, \beta) = M$$

These are all finite extensions $\Rightarrow M/F$ is finite $\Rightarrow \beta$ is alg. over F. So L/F is algebraic. ▨

$$\begin{array}{c} L \\ | \\ K \\ | \\ F \end{array}$$

<u>Consequences of</u> $[L:F] = [L:K][K:F]$

<u>Q</u>: Does one of $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt[3]{2})$ contain the other? (Both $\subseteq \mathbb{R}$)

<u>A</u>: No since $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$ and $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ and if $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[3]{2})$ then one has $2 \mid 3$ which is silly.

<u>Compositum</u> of fields $K_1, K_2 \subseteq L$ is the smallest subfield which contains both, and is denoted $K_1 K_2$.

Ex: $\mathbb{Q}(\sqrt{2})\mathbb{Q}(\sqrt[3]{2}) = \mathbb{Q}(\sqrt[6]{2})$

Reason 1: $\mathbb{Q}(\sqrt[6]{2})$ contains $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt[3]{2})$ and $\sqrt{2}/\sqrt[3]{2} = 2^{1/2}\cdot 2^{-1/3} = 2^{1/6} = \sqrt[6]{2}$.

Reason 2: Any field containing $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt[3]{2})$ must have $[K:\mathbb{Q}]$ divisible by 2 + 3

$\Rightarrow [K:\mathbb{Q}] \geq 6$. As $[\mathbb{Q}(\sqrt[6]{2}):\mathbb{Q}] = 6$, it must be the compositum.

Thm: $F \subseteq K_1, K_2 \subseteq L$, with $[K_i:F] < \infty$

$$[K_1K_2:F] \leq [K_1:F][K_2:F]$$

Note: Clear if both K_i/F are simple.

Pf: Let $\{\alpha_i\}$ be an F-basis for K_1, with $\alpha_1 = 1$
$\{\beta_j\}$ be an F-basis for K_2, with $\beta_1 = 1$

Claim: $K_1K_2 = \{\sum a_{ij}\alpha_i\beta_j \mid a_{ij} \in F\} = K$

Clearly $K_i \subseteq K \subseteq K_1K_2$ so the issue is whether K is a subfield.

It's closed under $+$, and also $\times$ since

$$(\alpha_i \beta_j)(\alpha_k \beta_\ell) = (\alpha_i \alpha_k)(\beta_j \beta_\ell) =$$
$$= (\Sigma\, a_i \alpha_i)(\Sigma\, b_j \beta_j) = \sum \underbrace{a_i b_j}_{\in F} \alpha_i \beta_j$$

What about mult. inverses?

Fix $\gamma \in K$. Consider $T : K \longrightarrow \overset{\text{F-vectorspace}}{K}$, $\delta \longmapsto \gamma\delta$, which is an F-linear transformation. As L is an int domain $\ker(T) = \{0\} \Rightarrow T$ is onto as $[K:F] < \infty$. In particular, $\exists\, \delta \in K$ with $T(\delta) = 1$, i.e. $\delta\gamma = 1 \Rightarrow \gamma^{-1} = \delta \in K$. So K is a subfield and hence $= K_1 K_2$