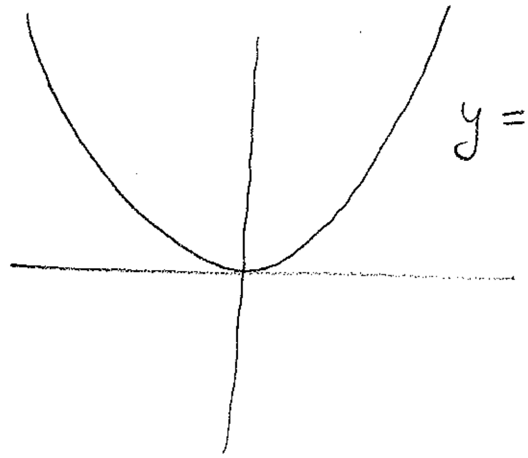
Lecture 39:

[Last time, talked about plane curves giving field extensions of $\mathbb{C}(t)$. Here's an example]



Ex: $V = \mathbb{V}(y - x^2)$

Consider $h(x,y) = y \in \mathbb{C}[V]$.

as a fn
$$V \longrightarrow \mathbb{C} = \{y\text{-axis}\}$$

Gives a ring homomorphism

$$\mathbb{C}[t] \xrightarrow{h^*} \mathbb{C}[V] \quad \text{via } h^*(f(t)) = f(h(x,y))$$
$$t \longmapsto y \qquad\qquad\qquad\qquad = f(y)$$

as h is non-constant, this gives a 1-1 field homom:

$$\mathbb{C}(t) \hookrightarrow \mathbb{C}(V)$$
$$t \longmapsto y$$

Let's identify $\mathbb{C}(t)$ with $\underbrace{\mathbb{C}(y)}_{\text{can read this either way...}} \subseteq \mathbb{C}(V)$,

and call it F.

Let $K = \mathbb{C}(V)$. I want to understand

K/F. Observe: ① K/F is simple, in particular $K = F(x)$

② x is alg. over F, being a root of $X^2 - t$.

③ $X^2 - t$ is irreducible.

So $K = F[z]/(z^2-t) = F(\sqrt{t})$

<u>Fun Fact</u>: As abstract fields, $F \cong K$ since if we had proj onto the x-axis, we would have found

$$\begin{aligned} \mathbb{C}(t) &\hookrightarrow K \\ t &\longrightarrow x \end{aligned}$$

and $K = \mathbb{C}(x)$

(in both senses).

In general, the same reasoning shows

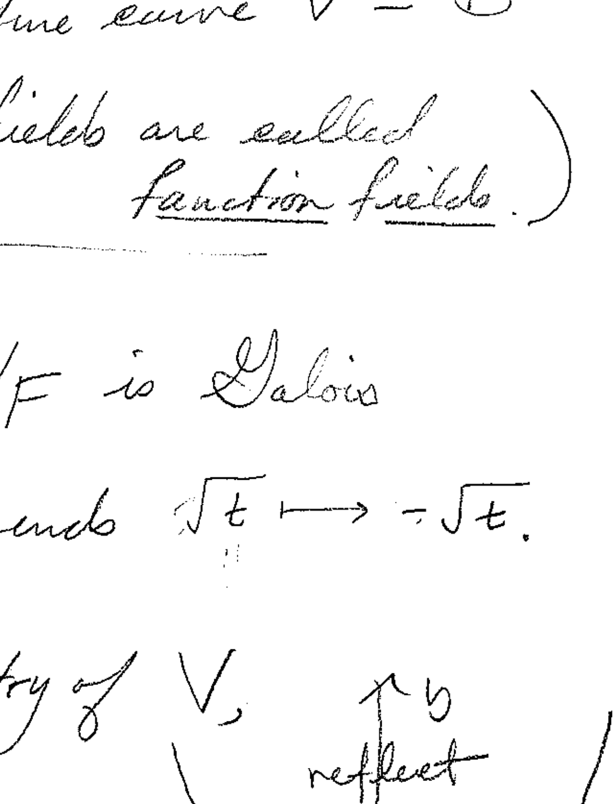
<u>Thm</u>: $V = \mathbb{V}(f) \subseteq \mathbb{C}^2$ an irreducible plane curve. Then $\mathbb{C}(V)$ is a finite extension of $\mathbb{C}(t)$.

This has a partial converse:

<u>Thm</u>: Suppose K is a finite extension of $\mathbb{C}(t)$. Then $\exists$ an irred, smooth affine curve $V \subseteq \mathbb{C}^n$ where $\mathbb{C}(V) = K$. (Such fields are called <u>function fields</u>.)

Back to the example: K/F is Galois with group $G = \mathbb{Z}_2$ which sends $\sqrt{t} \longmapsto -\sqrt{t}$.

This corresponds to a symmetry of V, namely $x \longrightarrow -x$.



Our general approach to the inverse Galois problem is reverse this process:

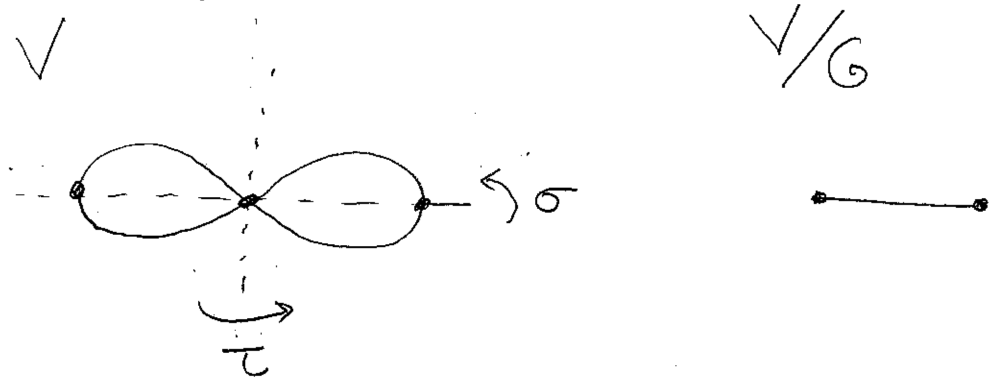
1. Given a finite group G, find a curve V (maybe in $\mathbb{C}^n$ or $\mathbb{P}^n_{\mathbb{C}}$) with G as a group of symmetries.

2. Each $\sigma \in G$ induces a field auto of $\mathbb{C}(V)$ [think as $\mathbb{C}(V)$ as fn on V]

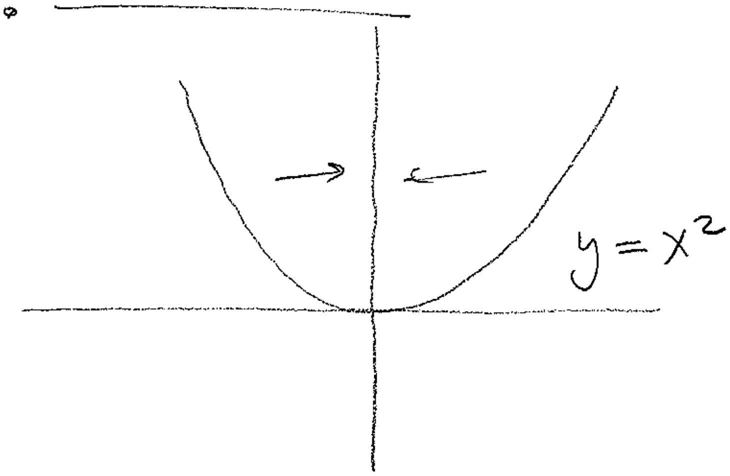
3. Identify $\mathbb{C}(V)_G$ with $\mathbb{C}(V/G)$ where V/G is the quotient of V/G, which is also an alg. curve.

4. Do ① so that $V/G = \mathbb{P}^1_{\mathbb{C}}$ and hence $\mathbb{C}(V/G) = \mathbb{C}(t)$. Thus, have built an ext. $\mathbb{C}(V)/\mathbb{C}(t)$ with Galois group G.

Thinking about ③:



$G = \mathbb{Z}_2 \times \mathbb{Z}_2$



<u>Back to the example</u>:

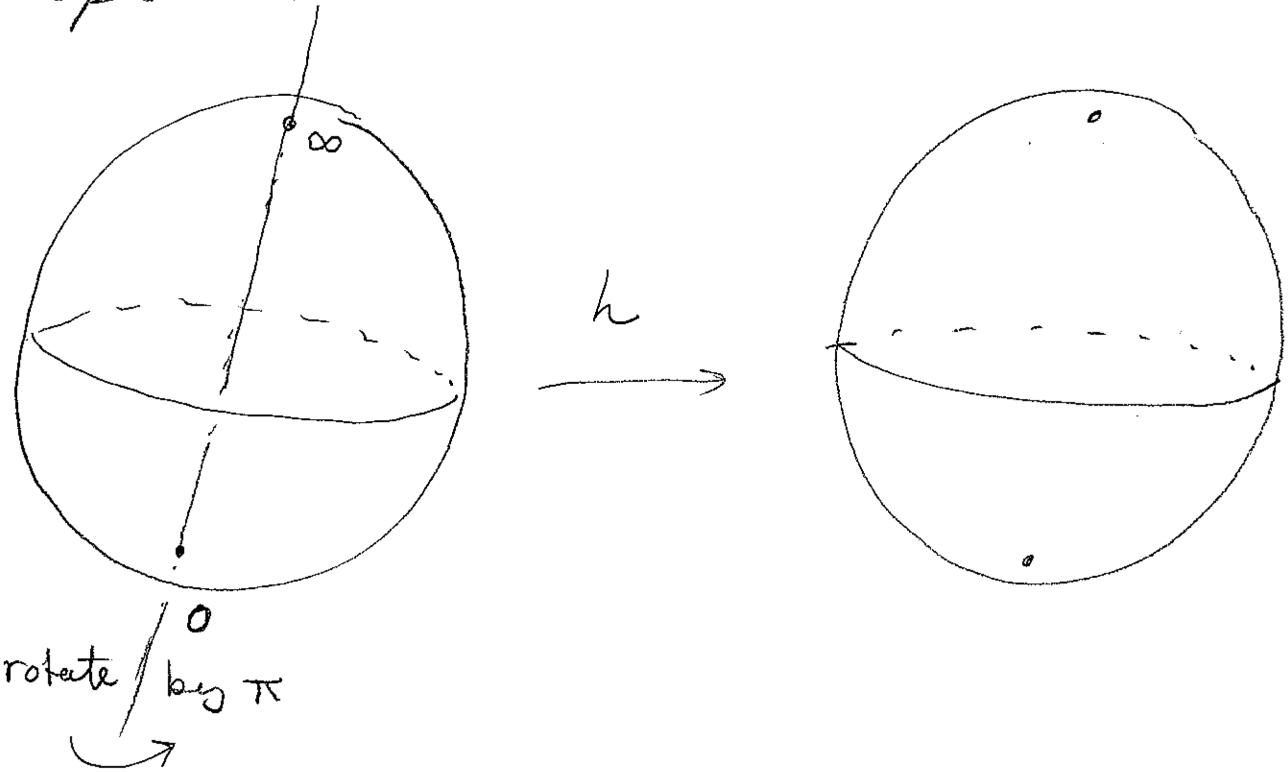
$$V \xrightarrow{h} \mathbb{C}$$
$$(x,y) \longrightarrow y$$

Now $V \cong \mathbb{C}$ via proj onto the x axis

The map $h : \mathbb{C} \longrightarrow \mathbb{C}$
$$z \longmapsto z^2$$

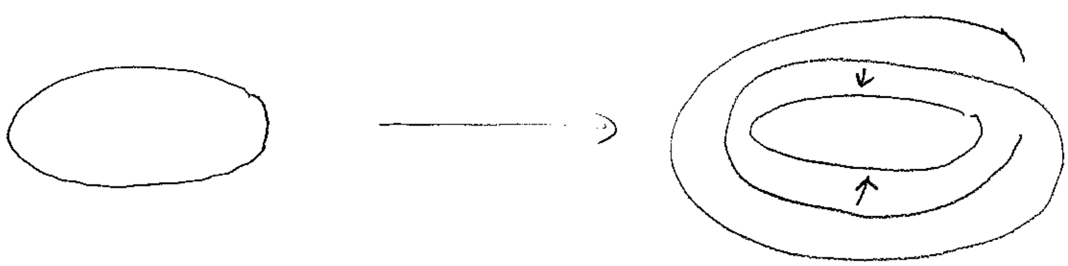
Let $\overline{V} \subseteq \mathbb{P}^2_{\mathbb{C}}$ be the corresponding proj curve. Have $\overline{V} \cong \mathbb{P}^1_{\mathbb{C}} =$  and so consider $\overline{h} : \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$
$$z \longmapsto z^2$$

In pictures



$z \mapsto -z$ or $(u:v) \to (-u:v)$

Map on the equator looks like



Here, h is a <u>branched cover</u>:

locally 1-1 except at a few pts

where it looks like $z \to z^4$.