Lecture 37:

Topology of plane curves in $\mathbb{P}^2_{\mathbb{C}}$:

$V = \mathbb{V}(f)$ where f = homog. poly in $\mathbb{C}[x,y,z]$

Assume V is smooth and irred.

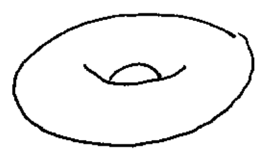
Examples we've seen:

① f linear, i.e. V is a line. By HW, all are the same, so can focus on

$V = \mathbb{V}(y) = \left(\begin{smallmatrix} y\text{-axis} \\ +\text{pt at } \infty \end{smallmatrix}\right) = \mathbb{P}^1_{\mathbb{C}} =$ 

② f quad, i.e. V = conic.

$V = \mathbb{P}^1_{\mathbb{C}} =$ 

③ f cubic, i.e. V = elliptic curve $=$ 
which has a group law.

In general, V is a compact orientable surface and is one of [g is called the genus of V]

$g = 0$ 1 2 3

While this is over $\mathbb{C}$, there are important consequences even when $k = \mathbb{Q}$.

Ex:

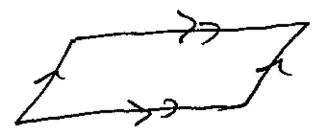
F.L.T. When $n \geq 3$,

$$\{\mathbb{V}_{\mathbb{P}^2_{\mathbb{Q}}}(x^n + y^n - z^n)\} = \emptyset$$

Suppose $f \in \mathbb{Q}[x,y,z]$ homogeneous. Consider

$$\left(V_{\mathbb{Q}} = \mathbb{V}_{\mathbb{P}^2_{\mathbb{Q}}}(f)\right) \subseteq \left(V_{\mathbb{C}} = \mathbb{V}_{\mathbb{P}^2_{\mathbb{C}}}(f)\right)$$

How many points $V_{\mathbb{Q}}$ has depends on the genus of $V_{\mathbb{C}}$.

genus	$V_{\mathbb{Q}}$	Symmetries of $V_{\mathbb{C}}$	Geometry of $V_{\mathbb{C}}$
0 	$\mathbb{P}^1_{\mathbb{Q}}$ <u>or</u> ϕ $x^2+y^2-z^2$ vs. $x^2+y^2-3z^2$	$PGL_2\mathbb{C} =$ $z \longmapsto \frac{az+b}{cz+d}$	round sphere  with unique shape
1 	$V_{\mathbb{Q}}$ is a finitely gen gp (Mordell-Weil) i.e. $\exists\, p_i \in V_{\mathbb{Q}}$ s.t. $V_{\mathbb{Q}} = \{n_1p_1 + \cdots + n_kp_k \mid n_i \in \mathbb{Z}\}$	trans. by group elts + finite gp	Euclidean torus  $\dim_{\mathbb{C}}\begin{pmatrix}\text{moduli}\\ \text{sp of Elliptic curves}\end{pmatrix} = 1$
≥ 2 	<u>Falting's Thm</u> (1980s) $V_{\mathbb{Q}}$ is finite (Almost gives FLT)	finite	Hyperbolic geom. $3g-3$ dim'l moduli space

Goal:

Thm: G a finite gp. Then $\exists$ a Galois extension ~~K~~ $K/\mathbb{C}(t)$ with group G.

So, need to associate a field with a variety somehow...

V affine alg variety $\subseteq k^n$.

$$k[V] = \{f : V \to k \mid f = \text{rest. of a poly}\}$$
$$= k[x_1, \dots, x_n] / \mathbb{I}(V)$$

If V is irreducible, then $k[V]$ is an ~~???~~ integral domain.

Def: The function field of an irred variety V, denoted $k(V)$, is the

the field of fractions of $k[V]$.

An elt of $k(V)$ is call a <u>rational function</u>

and is

$$f = \frac{g}{h} \quad \text{for polys } g, h \in k[x_1, \dots, x_n]$$

<u>Ex</u>: $k = \mathbb{C}$ and $V = \mathbb{C}$. Then

$$\mathbb{C}[V] = \mathbb{C}[t] \quad \text{and so}$$

$$\mathbb{C}(V) = \underset{\text{in } t}{\text{rat'l fns}} = \mathbb{C}(t)$$

[Notice connection to Goal Thm]

$$f = c\,\frac{(t-a_1)\cdots(t-a_k)}{(t-b_1)\cdots(t-b_\ell)} \quad \text{no } a_i = b_j, \; c \in \mathbb{C}.$$

Not quite a function $f: V \to \mathbb{C}$ as not defined at b_i.

<u>Def</u>: $f \in k(V)$ is <u>regular</u> at $p \in V$ if there is an expression $f = \frac{g}{h}$ where $h(p) \neq 0$.
Set

$$\mathrm{dom}(f) = \{p \in V \mid f \text{ reg at } p\}.$$

$\underline{\text{Ex}}$: $k = V = \mathbb{C}$, with $f \in \mathbb{C}(V)$ as above. Then

$$\operatorname{dom}(f) = \mathbb{C} \setminus \{b_1, \dots, b_e\}.$$

$\underline{\text{Ex}}$: $V = \mathbb{V}(xw - yz) \subseteq k^4$

Consider $f = \frac{x}{y} \in k(V)$. As $xw = yz$ in $k[V]$, another valid description of f is $f = \frac{z}{w}$. Thus

$$\operatorname{dom}(f) \supseteq \{\text{all pts of } V \text{ with } y \neq 0 \ \underline{\text{or}}\ w \neq 0\}$$

<u>Underlying point</u>: $k[V]$ is not a U.F.D.

Now focus on

$$V = \mathbb{V}(p) \subseteq \mathbb{C}^2$$

a smooth irreducible plane curve.

<u>Prop</u>: If $f \in \mathbb{C}(V)$ then

$$\mathrm{dom}(f) = V \setminus \{\text{finite set of pts}\}$$

<u>Pf</u>: $f = \frac{g}{h}$ for $g, h \in \mathbb{C}[V]$. Need $h(p) = 0$ for only finitely many points p.

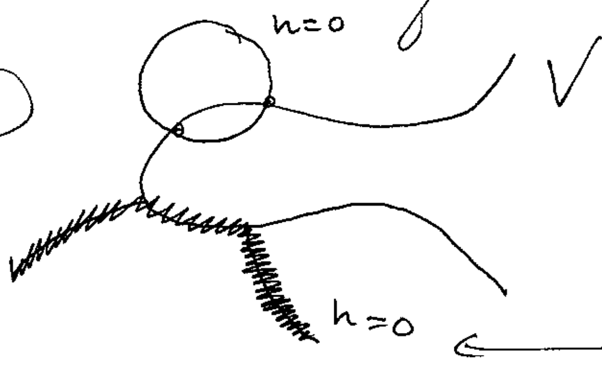
Equivalently

$$V' = \mathbb{V}(h, \underbrace{p}_{\text{def poly of } V}) = \text{finite set}.$$

Two approaches

(1) Dimension of a variety

(2)



Can't happen.