

Lecture 35: Projective space II.

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Last time:

$$\begin{aligned}\mathbb{P}_K^n &= \left\{ \text{all lines in } K^{n+1} \text{ through } 0 \right\} = \left\{ x \in K^{n+1} \mid x \neq 0 \right\} / \begin{array}{l} x \sim \lambda x \\ \text{for } \lambda \in K^* \end{array} \\ &= \left\{ (x_1 : \dots : x_n : 1) \right\} \cup \left\{ (x_1 : x_2 : \dots : x_n : 0) \right\} \\ &= K^n \cup \mathbb{P}_K^{n-1} \quad \left[\text{Example of a } \underline{\text{moduli space}} \right] \end{aligned}$$

Focus on $\mathbb{P}_R^2 = \frac{\text{circle}}{x \sim -x}$

Def: $f \in R[x, y, z]$ is homogenous if each term $d x^a y^b z^c$ has the same total degree $a+b+c$.

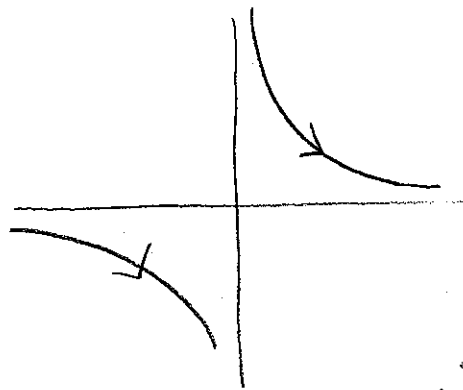
For such f , can define the projective variety.

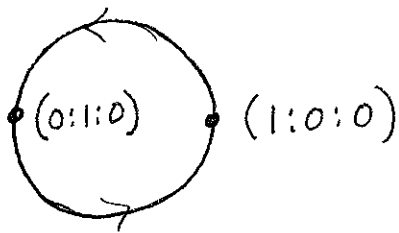
$$V(f) = \left\{ (a:b:c) \in \mathbb{P}_R^2 \mid f(a,b,c) = 0 \right\}$$

Ex: $V = V(xy - z^2)$

$$V \cap R^2 = V_{R^2}(xy - 1)$$

$$V \cap \mathbb{P}_R^1 = \left\{ (1:0:0), (0:1:0) \right\}$$





Will show that it is the
"same" as $V_{\mathbb{R}^2}(x^2+y^2-1)$.

$\mathbb{P}_{\mathbb{R}}^2$ is very symmetric: let $A \in GL_3 \mathbb{R}$,
i.e. A is a 3×3 matrix with $\det(A) \neq 0$.

Consider the projective transformation:

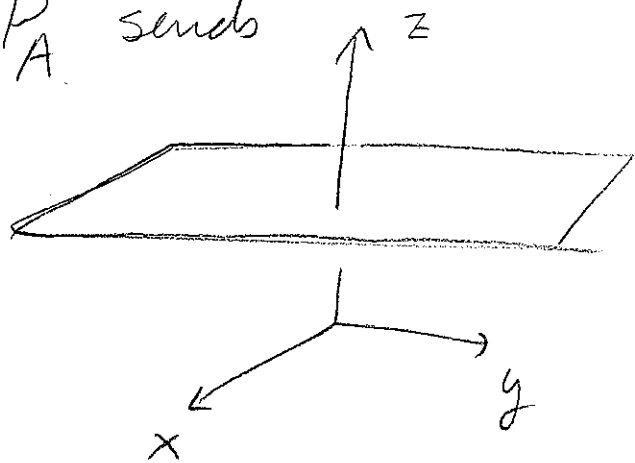
$$P_A: \mathbb{P}_{\mathbb{R}}^2 \rightarrow \mathbb{P}_{\mathbb{R}}^2 \quad \text{via } x \rightarrow Ax$$

which makes sense as $\lambda x \mapsto A(\lambda x) = \lambda(Ax)$

Ex: cly $A = \begin{pmatrix} B & 0 \\ 0 & 0 & 1 \end{pmatrix}$ then P_A sends

$\mathbb{R}^2 = \{(x:y:1)\}$ to itself,

and acts on it via the
linear trans w/ matrix B .



Ex: $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ gives translations.

$$P_A(x:y:1) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+2 \\ y+3 \\ 1 \end{pmatrix} = (x+2:y+3:1)$$

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Ex: $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $P_A(x:y:z) = (z:x:y)$

Takes $\mathbb{P}^1 = \{(x:y:0)\}$ to \swarrow y -axis in \mathbb{R}^2
 $\{(0:x:y)\} = \{(0:1:0)\} \cup \{(0:x:1)\}$

[Moral: Nothing special about \mathbb{P}^1 - it's just another line.]

Ex: $A = \begin{pmatrix} 0 & 0 & 1 \\ 1/2 & -1/2 & 0 \\ 1/2 & 1/2 & 0 \end{pmatrix}$ $V = \mathbb{V}(\overbrace{xy - z^2}^{f \in \mathbb{R}[x,y,z]})$

Q: What is $P_A(V)$? [Should be another variety...]

Consider
$$\begin{array}{ccccc} \mathbb{P}_{\mathbb{R}}^2 & \xleftarrow{P_A} & \mathbb{P}_{\mathbb{R}}^2 & \xrightarrow{f} & \mathbb{R} \\ \cup & & \cup & & \\ P_A(V) & & V & & \end{array}$$

and so $P_A(V) = \mathbb{V}(f \circ P_A^{-1})$. Use coord

$$\begin{array}{ccc} \mathbb{P}_{\mathbb{R}}^2 & \xrightarrow{P_A^{-1}} & \mathbb{P}_{\mathbb{R}}^2 \\ (u:v:w) & & (x,y,z) \end{array}$$

So

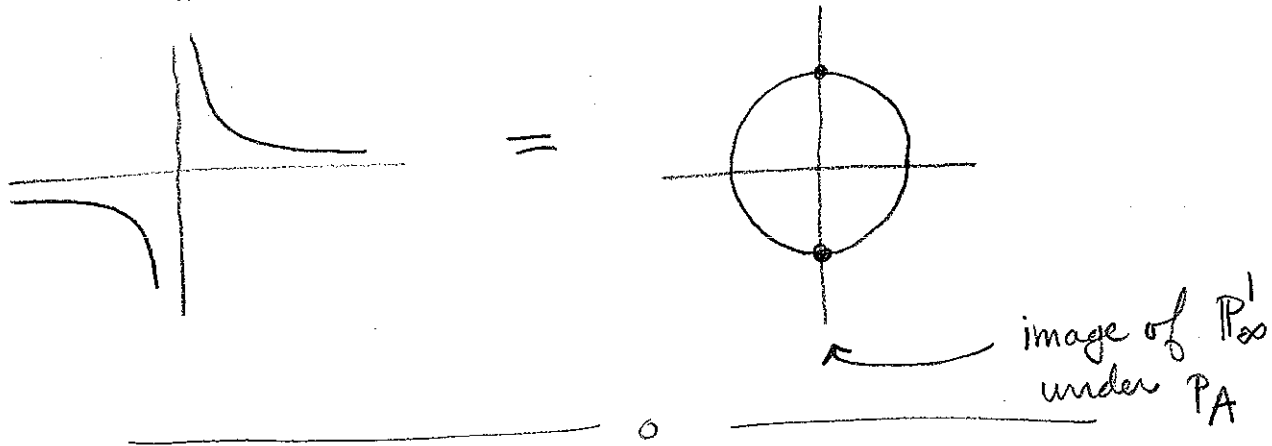
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v+w \\ -v+w \\ u \end{pmatrix}$$

and $xy - z^2 = (v+w)(-v+w) - u^2 = -v^2 + w^2 - u^2$

$$V' = P_A(V) = \mathbb{V}_{\mathbb{P}^2} (u^2 + v^2 - w^2) = \mathbb{V}_{\mathbb{R}^2} (u^2 + v^2 - 1)$$

no pts at ∞

So in $\mathbb{P}^2_{\mathbb{R}}$, have



General Conic: In \mathbb{R}^2 , consider variety of

$$g(x,y) = ax^2 + 2bxy + cy^2 + 2dx + 2ey + f$$

which comes from

$$V = \mathbb{V}_{\mathbb{P}^2} (ax^2 + 2bxy + cy^2 + 2dxz + 2eyz + fz^2)$$

homogenisation

Have

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$$q(x, y, z) = (x \ y \ z) \overbrace{\begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix}}^M \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

which is a quadratic form. There $\exists A \in GL_3 \mathbb{R}$ such that $A^T M A = \begin{pmatrix} \varepsilon_1 & & 0 \\ & \varepsilon_2 & \\ 0 & & \varepsilon_3 \end{pmatrix}$ with $\varepsilon_i \in \{-1, 0, 1\}$

[Query: Why? cf. mess. discuss at length.]

So $V' = P_{A^{-1}}(V)$ is given by

$$\begin{aligned} q'(u, v, w) &= \left(A \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right)^T M \left(A \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right) = (u \ v \ w) \begin{pmatrix} \varepsilon_1 & & 0 \\ & \varepsilon_2 & \\ 0 & & \varepsilon_3 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \\ &= \varepsilon_1 u^2 + \varepsilon_2 v^2 + \varepsilon_3 w^2 \end{aligned}$$

Thm: Up to a proj. transformation, any conic in $\mathbb{P}_{\mathbb{R}}^2$ is one of

- (a) $V(x^2 + y^2 - z^2)$ (non-degenerate conic)
- (b) $V(x^2 + y^2 + z^2) = \emptyset$
- (c) $V(x^2 - y^2) = \boxed{HW}$
- (d) $V(x^2 + y^2) = \{(0:0:1)\}$
- (e) $V(x^2) = y\text{-axis}$
- (f) $V(0) = \mathbb{P}_{\mathbb{R}}^2$.

Let $C =$ nondegen conic.

$$L_{\infty} = \text{line at } \infty = \{(x:y:0)\}$$

Three cases:

$$C \cap L = \emptyset \implies \text{Ellipse}$$

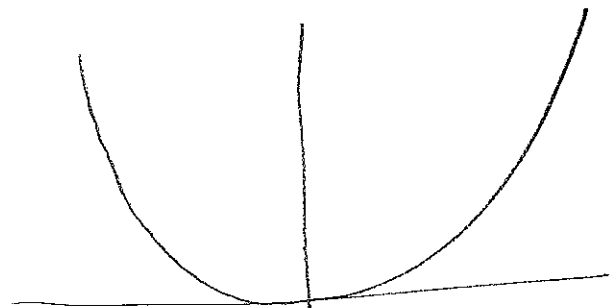
$$C \cap L = 2 \text{ pts} \implies \text{Hyperbola}$$

$$C \cap L = 1 \text{ pt, tangent} \implies \text{parabola.}$$

intersection

$$V(x^2 - y) =$$

\cap



$$V_{\mathbb{P}^2}(x^2 - yz) \longrightarrow V_{\mathbb{P}^2}(v^2 - wu) = V'$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

