Math 418: Takehome Midterm 2

Due date: In class on Wednesday, April 21.

Disclaimer, Terms, and Conditions: You may not discuss the exam with anyone except myself. You may *only* consult the following:

- The beloved(?) text, Dummit and Foote's *Abstract Algebra*.
- Cox, Little, and O'Shea, *Ideals, Varieties, and Algorithms*.
- Your class notes and returned HW sets.
- My online class notes and HW solutions.

You can use any result in chapters 14-15 of Dummit and Foote, even if I didn't cover it in class. You can also use the result of any HW problem that was assigned, whether or not you did it. While I believe all the questions are stated correctly, please contact me if you think something is fishy.

Office hours: Friday 3-4:30, Monday 10-11 and 3-5. No office hours on Tuesday.

1. Consider the cyclotomic field $K = \mathbb{Q}(\zeta_n)$ and let $G = \mathrm{Gal}(K/\mathbb{Q})$.

   (a) Let $\tau \in G$ be the restriction of complex conjugation. Find the element that τ corresponds to under the isomorphism $G \cong (\mathbb{Z}/n\mathbb{Z})^\times$.

   (b) Let $K^+ = K \cap \mathbb{R}$. Prove that $K^+ = \mathbb{Q}(\alpha)$ where $\alpha = \zeta_n + \zeta_n^{-1}$.

2. Find the Galois group of $x^4 - 7$ over $\mathbb{Q}$ explicitly as a permutation group on the roots

   $$\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} = \left\{\sqrt[4]{7}, -\sqrt[4]{7}, \sqrt[4]{7}i, -\sqrt[4]{7}i\right\}$$

   Clarification: Your answer should both give the isomorphism type of $\mathrm{Gal}(K/Q)$ and identify the explicit subgroup of $S_4 = \mathrm{SymmetricGroup}\big(\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}\big)$ which is its image under the map $\mathrm{Gal}(K/Q) \to S_4$.

3. In a field F *of characteristic 2*, consider an irreducible cubic $f(x) = x^3 + ax^2 + bx + c \in F[x]$.

   (a) Prove that the discriminant D of f is always a square in F.

   (b) Let K be the splitting field of f over F. Prove that K is also the splitting field of an irreducible polynomial of the form $x^3 + px + q \in F[x]$.

   (c) For $f(x) = x^3 + px + q$, consider the polynomial $g(x) = x^2 + qx + (p^3 + q^2)$. Prove that $\mathrm{Gal}(K/F)$ is A_3 if $g(x)$ has a root in F and S_3 otherwise.

4. For a field k, here are some basic problems for varieties in k^2, where we take the coordinates to be x, y. Except for part (b), *assume that k is algebraically closed.*

   (a) Let V be the x-axis, i.e. $V = \mathbb{V}(y)$. Prove that V is irreducible. Hint: Show a prime ideal is radical.

(b) Give an example of a field k, necessarily not algebraically closed, for which the x-axis is *reducible*.

(c) Prove that $V = \mathbb{V}(x - y)$ is irreducible.

(d) Prove that $S = \left\{(a, a) \in k^2 \mid a \neq 1\right\}$ is *not* an algebraic variety if $k = \mathbb{C}$.

(e) What is the decomposition of $V = \mathbb{V}(x^2 - y^2)$ into irreducibles? **Warning:** The answer depends on k!

5. Let V be an algebraic variety in k^n, and set $I = \mathbb{I}(V)$. Recall the coordinate ring of V is

$$\begin{aligned} k[V] &= \left\{f\colon V \to k \;\middle|\; f \text{ is the restriction of a polynomial in } k[x_1, \dots, x_n].\right\} \\ &= k[x_1, \dots, x_n]/I \end{aligned}$$

In particular, two elements of $k[V]$ are the same if they agree at every point of V, even if nominally they come from different polynomials.

The ring $k[V]$ contains k as the subring of constant functions, coming from the polynomials with only a constant term. Thus it is a vector space over k. Prove that the following are equivalent.

(a) V is a finite set of points in k^n.

(b) $k[V]$ is finite-dimensional as a k-vector space.

Hints: For (a) ⇒ (b) consider $k[V]$ as a subspace of the vector space of *all* functions $f\colon V \to k$. For (b) ⇒ (a), for each coordinate, try to show that the set $\{a_i \mid (a_1, \dots, a_n) \in V\}$ is finite.