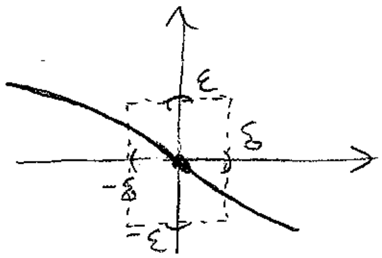
Lecture 7: Limits in several variables (Sect 14.2)

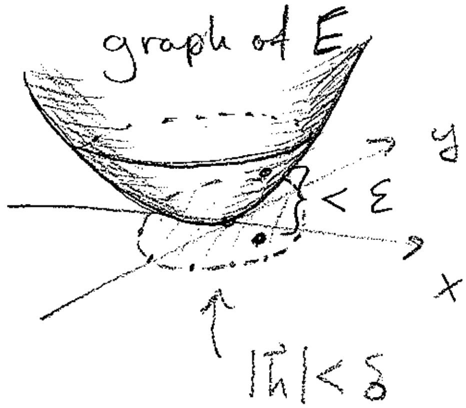
Last time: $E: \mathbb{R} \to \mathbb{R}$

Say $\lim_{h \to 0} E(h) = 0$ if given $\varepsilon > 0$ can always find a $\delta > 0$ so that when $0 < |h| < \delta$ then $|E(h)| < \varepsilon$



[Foreshadow story of the Sleipner A sinking.]

Now suppose $E: \mathbb{R}^2 \to \mathbb{R}$. We say $\lim_{\vec{h} \to 0} E(\vec{h}) = 0$ if given $\varepsilon > 0$ can always find $\delta > 0$ so that when $0 < |\vec{h}| < \delta$ then $|E(\vec{h})| < \varepsilon$.

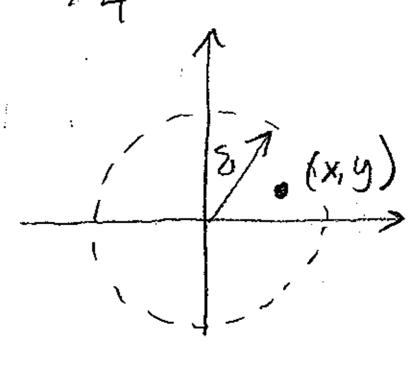


Ex:

$$\lim_{(x,y) \to (0,0)} x + 3y = 0$$

Reason: Let $\varepsilon > 0$ be given. Take $\delta = \varepsilon/4$.

If $h = (x, y)$ and $|h| < \delta$, then $|x| < \delta$ and $|y| < \delta$ as shown:



Then

$$|x+3y| \leq |x| + |3y| < \delta + 3\delta = 4\delta = \varepsilon.$$

More generally, if $f: \mathbb{R}^2 \to \mathbb{R}$ then

$$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = c \quad \text{if} \quad f(\vec{a}+\vec{h}) = c + E(\vec{h})$$

where $\lim_{\vec{h} \to \vec{0}} E(\vec{h}) = 0.$

Same notion of limit works for $f: \mathbb{R}^3 \to \mathbb{R}$ or $\mathbb{R}^n \to \mathbb{R}$ or even $\mathbb{R}^n \to \mathbb{R}^m$.

<u>A more complicated example</u>:

Take $f(x,y) = \dfrac{2xy}{x^2+y^2}$ ← not defined at $(0,0)$.

What is

$$\lim_{(x,y) \to \vec{0}} f(x,y) = ?$$

[First, try our usual trick: reduce the dimension]

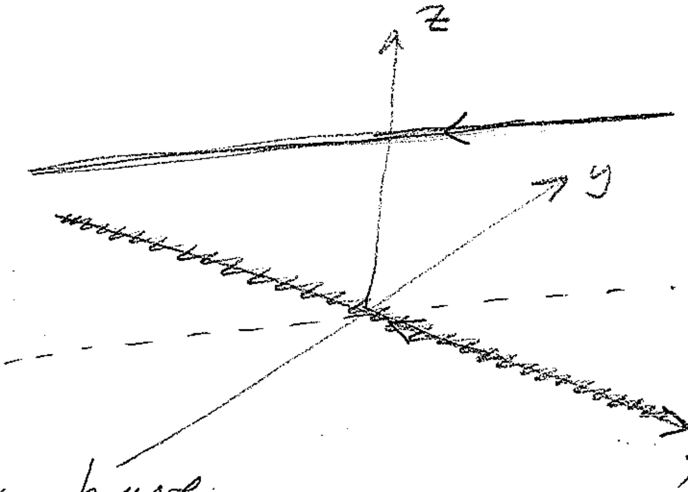
Along the x-axis:

$$f(x,0) = \frac{2x \cdot 0}{x^2+0^2} = 0$$

which suggests $\lim = 0$. But along the line $y = x$

$$f(x,x) = \frac{2x \cdot x}{x^2+x^2} = 1 \quad (\text{for } x \neq 0)$$

Thus the limit does not exist.



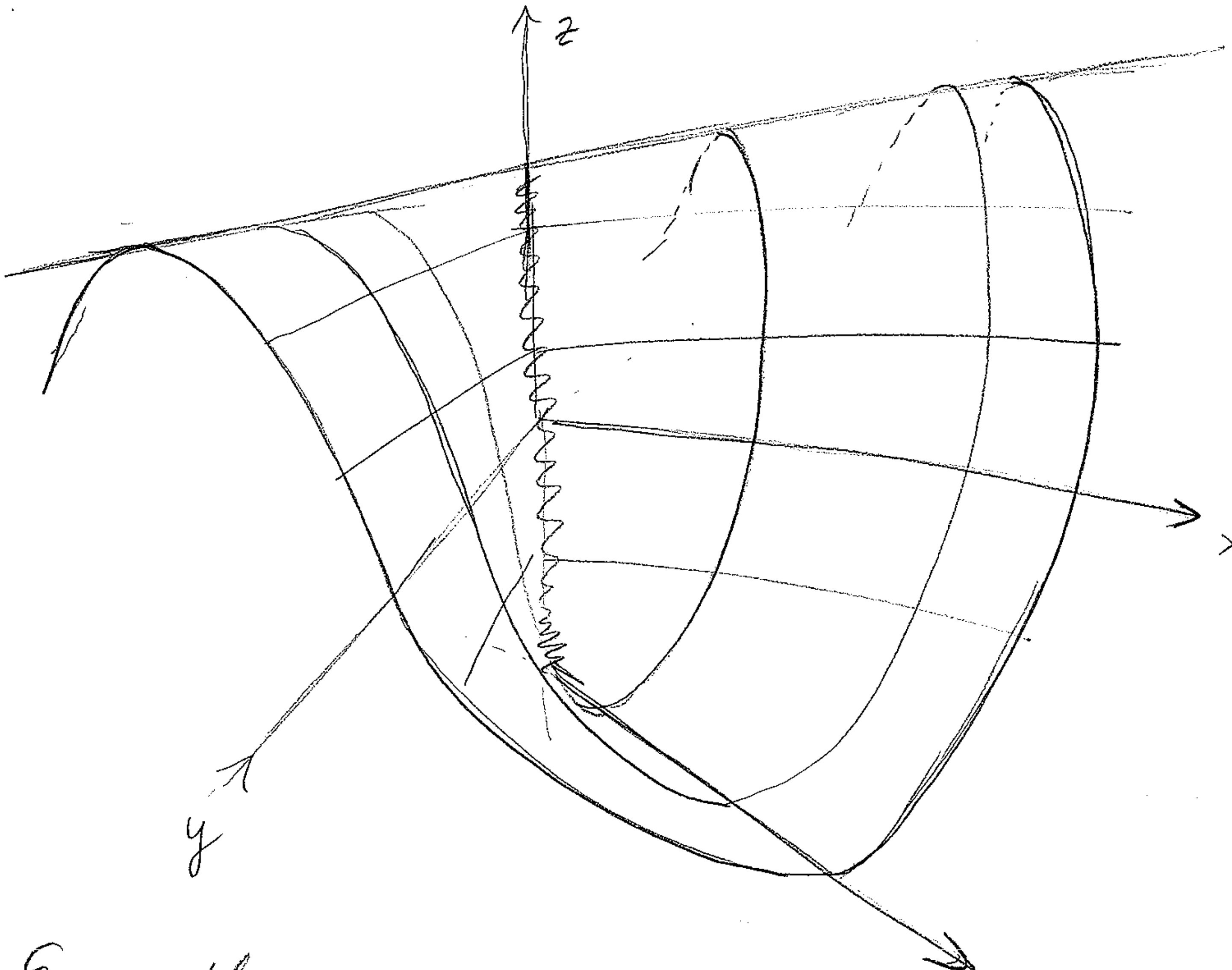
Can actually work out the graph of f.

Along the line $y = cx$ have

$$f(x,cx) = \frac{2x(cx)}{x^2+(cx)^2} = \frac{2cx^2}{(1+c^2)x^2} = \frac{2c}{1+c^2}$$

E.g. $f = -1$ along $y = -x$

The full picture is:



Even odder:

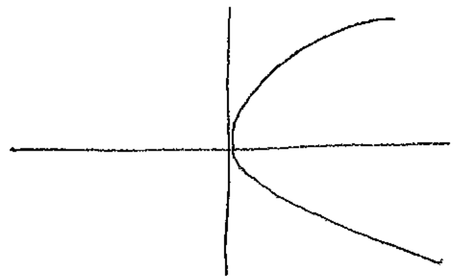
$$f(x,y) = \frac{xy^2}{x^2+y^4}$$ What is limit as $(x,y) \to \vec{0}$?

Along the line $y = cx$ we have

$$f(x,cx) = \frac{x(cx)^2}{x^2+(cx)^4} = \frac{c\,x^3}{x^2(1+c^4x^2)} = \frac{cx}{1+c^4x^2}$$

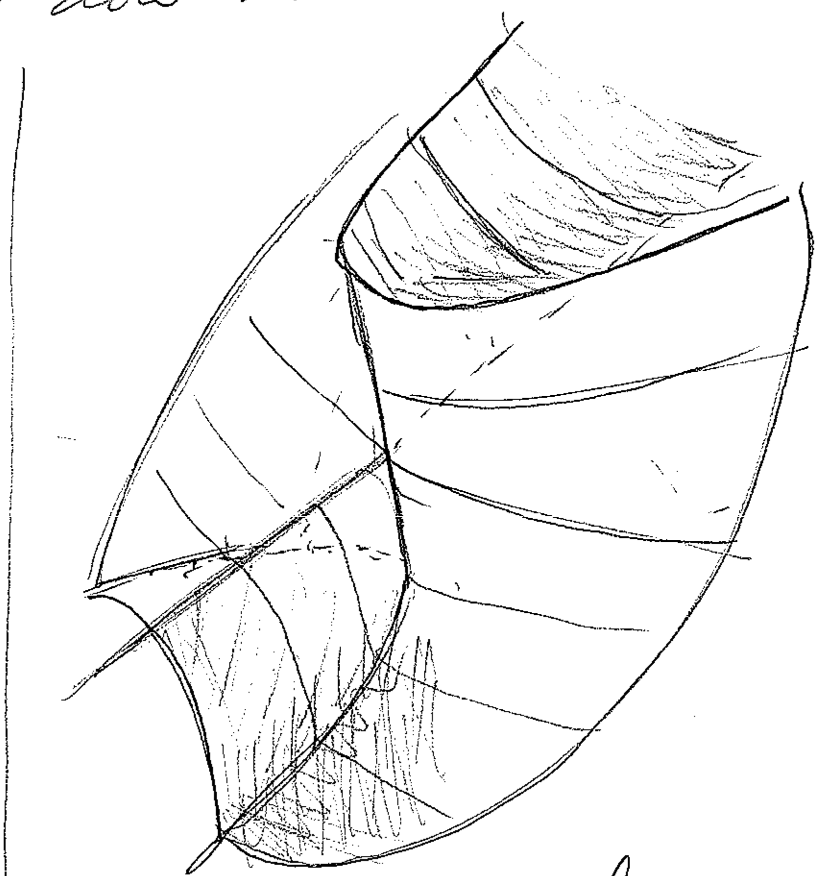
which $\to 0$ as $x \to 0$

But: Along $x = y^2$ we have



$$f(y^2, y) = \frac{y^2 \cdot y^2}{(y^2)^2 + y^4} = \frac{y^4}{2y^4} = 1/2.$$

So again, the limit does not exist!



Graph of f

Rules for limits:

$f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$

(a) $$\lim_{\vec{x} \to \vec{a}} \left(f(\vec{x}) + g(\vec{x})\right) = \lim_{\vec{x} \to \vec{a}} f(\vec{x}) + \lim_{\vec{x} \to \vec{a}} g(\vec{x})$$

(b) $$\lim_{\vec{x} \to \vec{a}} f(\vec{x}) g(\vec{x}) = \left(\lim_{\vec{x} \to \vec{a}} f(\vec{x})\right)\left(\lim_{\vec{x} \to \vec{a}} g(\vec{x})\right)$$

In both cases, this is provided the RHS all makes sense.

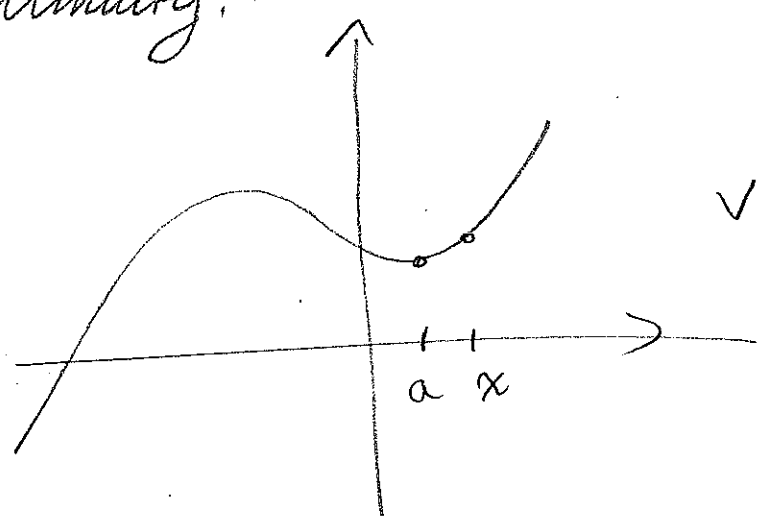
$$\lim_{\vec{x}\to\vec{a}} \frac{1}{f(\vec{x})} = \frac{1}{\lim_{x\to\vec{a}} f(\vec{x})}$$ provided is $\neq 0$.

Ex:

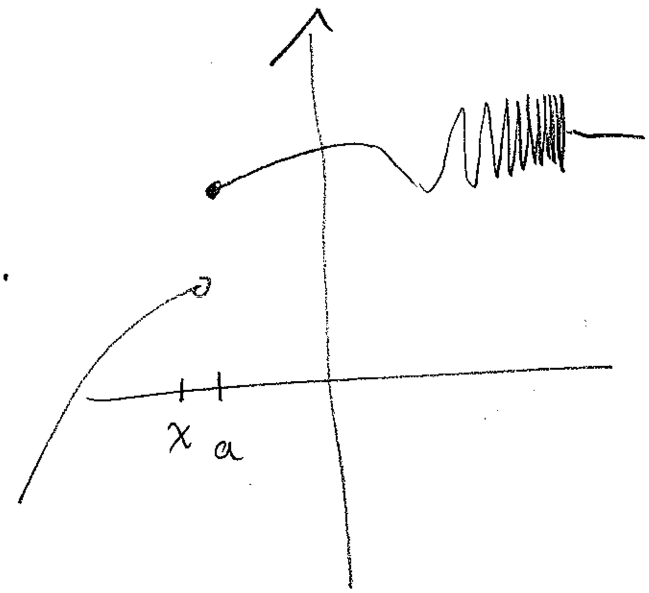
$$\lim_{\vec{x}\to(1,2)} \frac{x^2+2x}{y^2} = \frac{3}{4}.$$

Steps omitted

Continuity:



vs.



$f: \mathbb{R} \to \mathbb{R}$ is continuous at a if $\lim_{x\to a} f(x) = f(a)$. Similarly:

$f: \mathbb{R}^n \to \mathbb{R}$ is continuous at $\vec{a}$ if $\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = f(\vec{a})$.

Ex: $f(x,y) = \begin{cases} \dfrac{x^2}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

is continuous at $(0,0)$ since

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Reason: Given $\varepsilon > 0$, take $\delta = \varepsilon$.
If $\vec{h} = (x,y)$ has $0 < |\vec{h}| < \delta$ then

$$|f(x,y)| = \frac{|x^2|}{|\vec{h}|} \leq \frac{|\vec{h}|^2}{|\vec{h}|} = |\vec{h}| < \delta = \varepsilon.$$

This function f is also continuous at all other points of $\mathbb{R}^2$ since it is built up out of cont. pieces.