Lecture 15: More on min/max (14.7)

Last time: $f: \mathbb{R}^2 \to \mathbb{R}$ with (a,b) a crit pt.

$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

If $D > 0$ and

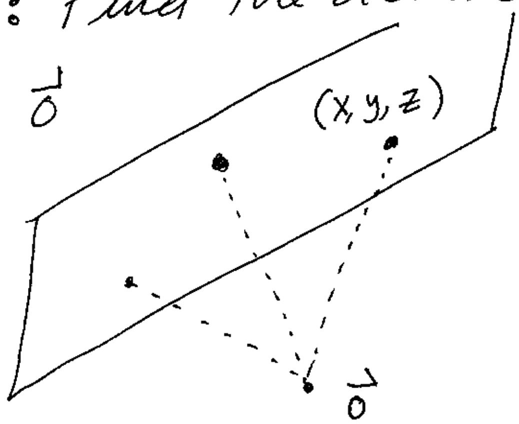
$f_{xx}(a,b) > 0 \Rightarrow$ Local min

$f_{xx}(a,b) < 0 \Rightarrow$ Local max

If $D < 0 \Rightarrow$ saddle.

> Similar test for $f: \mathbb{R}^n \to \mathbb{R}$ but need to talk about eigenvalues of matrices...

Ex: Find the distance from the plane $x - y + 2z = 3$ to $\vec{0}$



$$z = \frac{3 - x + y}{2}$$

WANT TO MINIMIZE:

$$f(x,y) = \left(\begin{array}{c}\text{dist from} \\ \left(x, y, \frac{3-x+y}{2}\right) \\ \text{to } \vec{0}\end{array}\right)^2$$

$$= x^2 + y^2 + \frac{1}{4}(3 - x + y)^2$$

Critical Points:

$\nabla f = \vec{0}$:

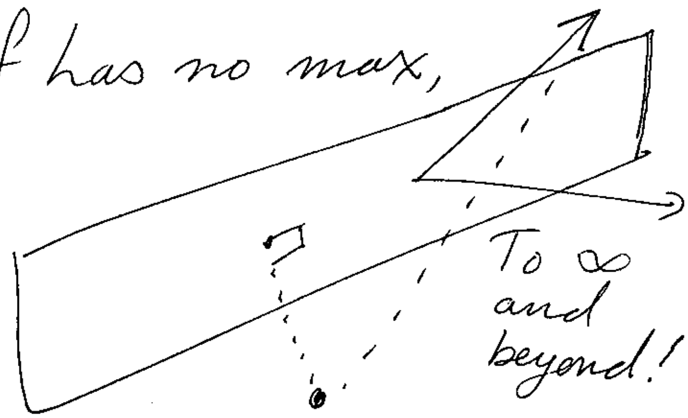
$$\frac{\partial f}{\partial x} = 2x + \frac{1}{2}(3 - x + y)\cdot(-1) = \frac{5}{2}x - \frac{1}{2}y - \frac{3}{2}$$

$$\frac{\partial f}{\partial y} = 2y + \frac{1}{2}(3 - x + y) = -\frac{1}{2}x + \frac{5}{2}y + \frac{3}{2}$$

Setting $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ we find a unique solution $x = 1/2$ and $y = -1/2$. Since there is only one crit. point, this must be the minimum and so the distance is

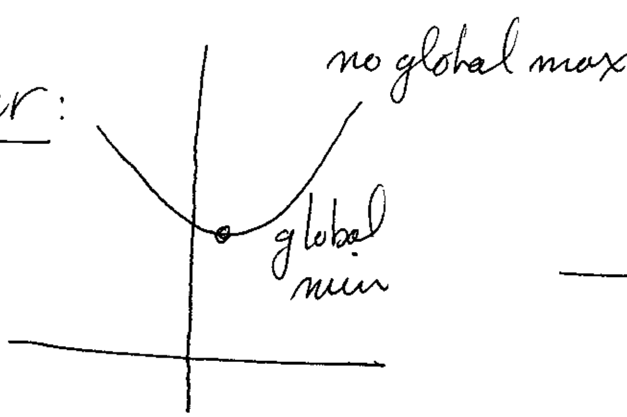
$$\sqrt{f(1/2, -1/2)} = \sqrt{\tfrac{1}{2}^2 + \left(-\tfrac{1}{2}\right)^2 + \tfrac{1}{4}(3-1)^2} = \sqrt{\tfrac{3}{2}}$$

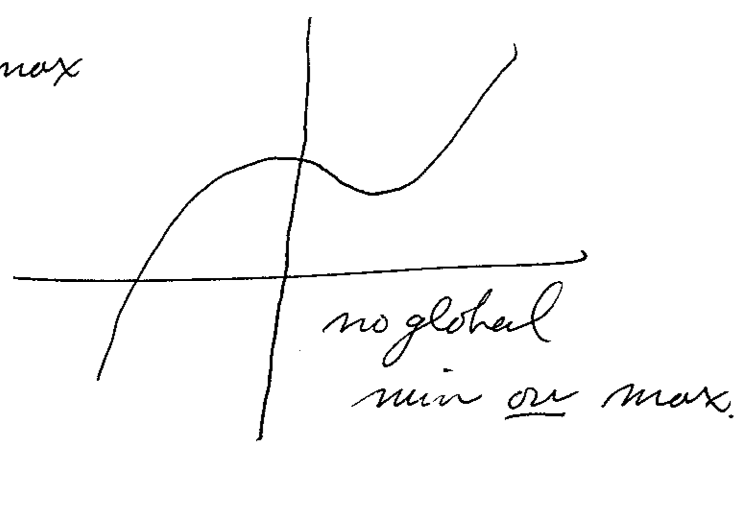
Hey, doesn't the same reasoning say that this is also the <u>maximum</u> dist? (as there's only one crit pt.) In fact, f has no max, as is clear geometrically:



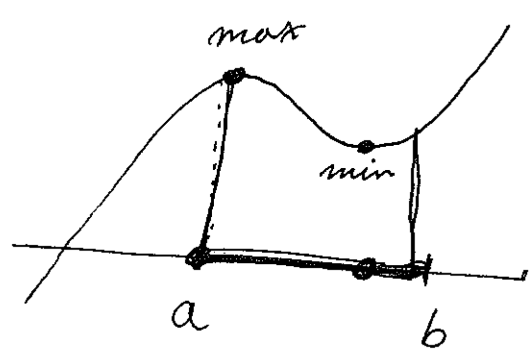
Also, it looks like there is a min, but lets look at gen. criteria for mins/maxes to exist...

<u>One var</u>:





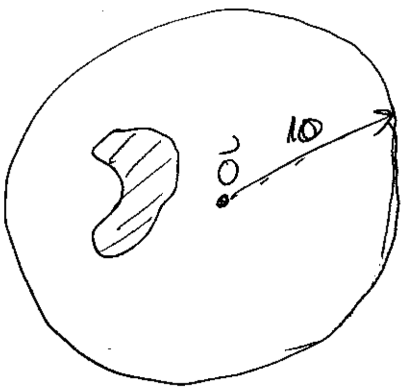
Extreme Value Theorem: f continuous on $[a,b] = \{a \le x \le b\}$. Then f has both a global min and max.



These min/max occur at either
a) a crit pt of f b) one of a or b.

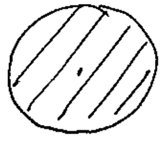
Two Var: D subset of $\mathbb{R}^2$

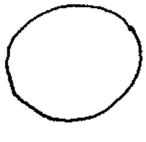
Bounded: Contained within a disk:

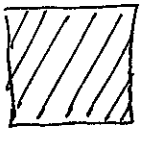


Closed: Contains all its boundary points.

Closed:

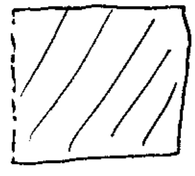
$|\vec{x}| \le 1$ 

$|\vec{x}| = 1$ 

$\left\{ \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le 1 \end{matrix} \right\}$ 

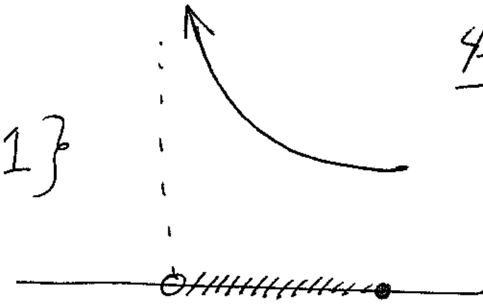
Not Closed:

$|\vec{x}| < 1$ 

 $\left\{ \begin{matrix} 0 < x \le 1 \\ 0 \le y \le 1 \end{matrix} \right\}$

Closed important in 1 var too:

$$f = \frac{1}{x} \text{ on } (0,1] = \{0 < x \leq 1\}$$



<u>Extreme Value Theorem</u>: f continuous on D in $\mathbb{R}^n$.

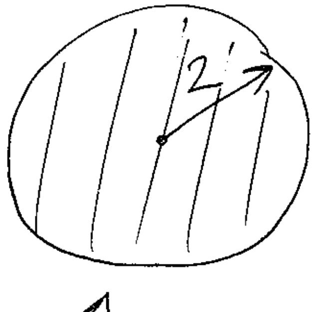
If D is closed and bounded, then it has a global min and max. These occur either at crit pts of f or one the boundary of D.

Back to original prob:

$$f(x,y) = x^2 + y^2 + \frac{1}{4}(3 - x + y)^2$$

with one crit pt at $(1/2, -1/2)$.

Consider $D = \{ |\vec{x}| \leq 2 \}$. On D, f has one crit pt where $f = 3/2$



∂D

On ∂D, $f(x,y) \geq x^2 + y^2 = 4$.

So $3/2$ is the global min value of f on D.

As $f(x,y) \geq \cancel{4}$ outside D, $3/2$ is also the global min on all of $\mathbb{R}^2$.

<u>Double Check</u>: $f_{xx} = 5/2 \quad f_{xy} = -1/2 \quad f_{yy} = 5/2$

$$D = \begin{vmatrix} 5/2 & -1/2 \\ -1/2 & 5/2 \end{vmatrix} = \frac{25}{4} - \frac{1}{4} = 6 > 0 \text{ and } f_{xx} > 0.$$

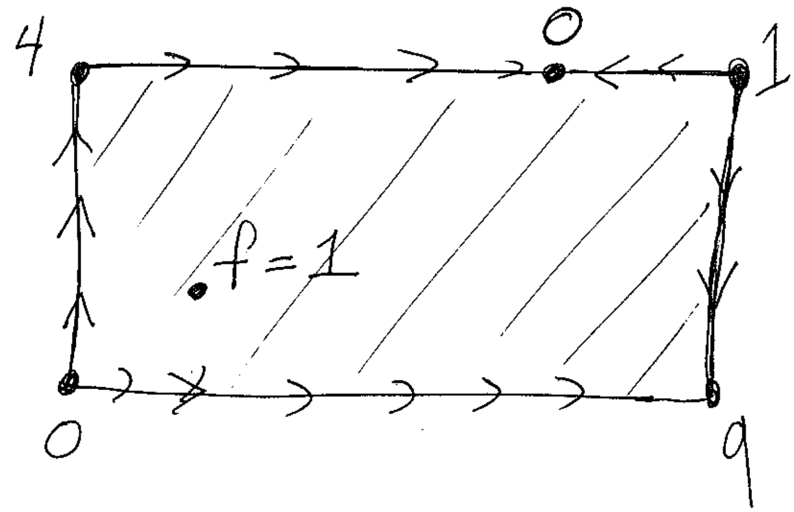
<u>Ex</u>: Find the global min/max of

$$f(x,y) = x^2 - 2xy + 2y \quad \text{on} \quad D = \left\{ \begin{matrix} 0 \leq x \leq 3 \\ 0 \leq y \leq 2 \end{matrix} \right\}$$

<u>Crit pts</u>: $f_x = 2x - 2y$

$f_y = -2x + 2$

$\nabla f = \vec{0} \implies x = 1,\ y = 1$



<u>On Left</u>: $f(0,y) = 2y$ min at 0: $f = 0$
max at 2: $f = 4$

<u>On Right</u>: $f(3,y) = 9 - 6y + 2y = 9 - 4y$

min at $y = 2$: $f = 1$

max at $y = 0$: $f = 9$

On top: $f(x,2) = x^2 - 4x + 4$

at 0, f is 4 at 3, f is 1

Also, there's a crit pt $f_x = 0$ at $x = 2$

and $f = 0$ there

On bottom: $f(x, 0) = x^2$; min at 0, max at 3.

So: Max on boundary: 9
Min on boundary: 0
} global min / max.

Value at crit pt: 1