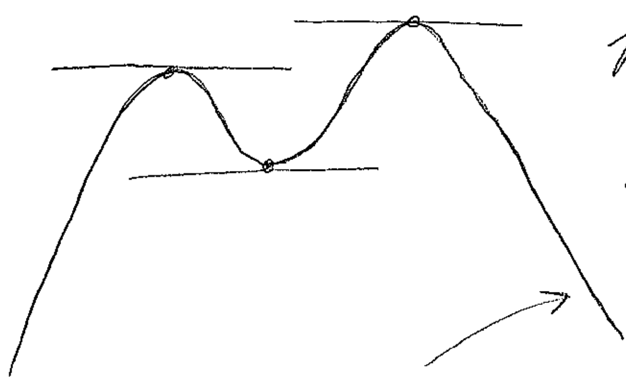
Lecture 14: Intro to min/max (14.7)

One Var:

Min/max occur at critical points where $f'(x) = 0$



no global min

Local vs. Global

2^{nd} derivative test

$f''(x) < 0$: max

$f''(x) > 0$: min

Need to do the same for $f: \mathbb{R}^n \to \mathbb{R}$, focus on $f: \mathbb{R}^2 \to \mathbb{R}$ first.

Critical Points: $\nabla f(\vec{p}) = \vec{0}$

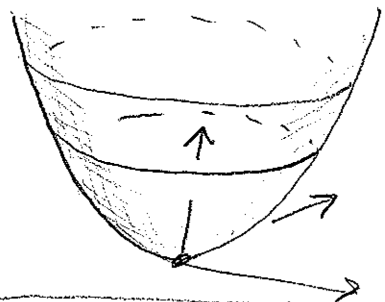
Every local extema happens at a critical point.

Today we'll talk about the 2^{nd} derivate test for f.

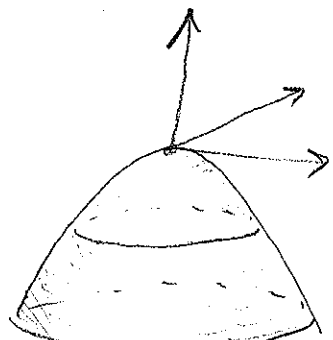
Three critical points at $(0,0)$:

$f(x,y) = x^2+y^2$ $\quad$ $f(x,y) = -x^2-y^2$ $\quad$ $f(x,y) = x^2-y^2$

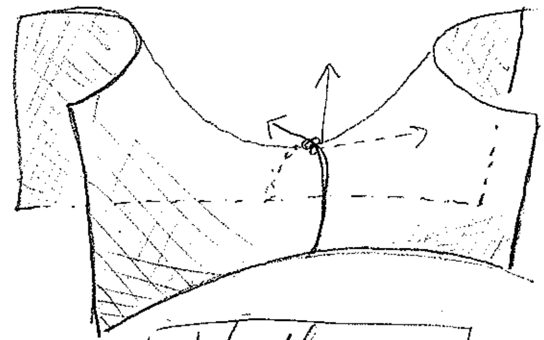
$\nabla f = (2x, 2y)$ $\quad$ $\nabla f = (-2x,-2y)$ $\quad$ $\nabla f = (2x, -2y)$



Local Min

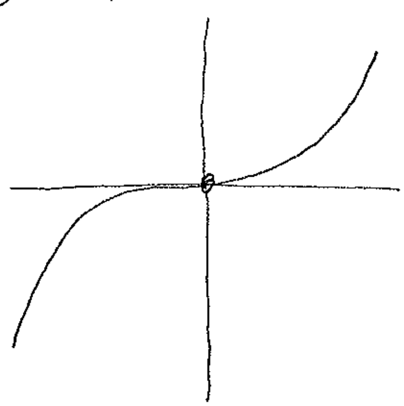


Local Max



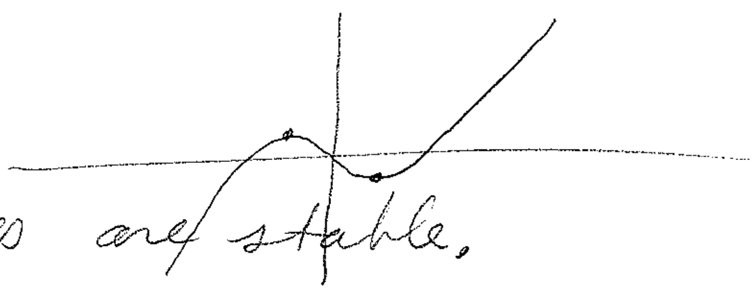
Neither: Saddle

In one var, also a neither case,
e.g. $f(x) = x^3$



$$\left.\begin{aligned} f'(x) &= 3x^2 \\ f''(x) &= 6x \end{aligned}\right\} \text{ both 0 at 0.}$$

This was pretty rare, because its not "stable." E.g. $f(x) = x^3 + \frac{1}{10}x$ doesn't have this issue



However, saddles are stable.

One pt of view on the 2nd der. test:

Taylor series at x_0

$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + E(h)$$

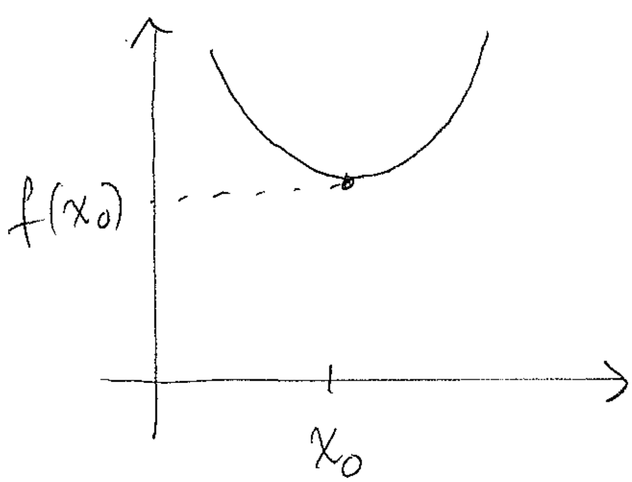
where $E(h)$ is really small, e.g. $\lim_{h\to 0} \frac{E(h)}{h^2} = 0$.

If x_0 is a critical point, then

$$f(x_0+h) = f(x_0) + \frac{f''(x_0)}{2}h^2 + E(h)$$

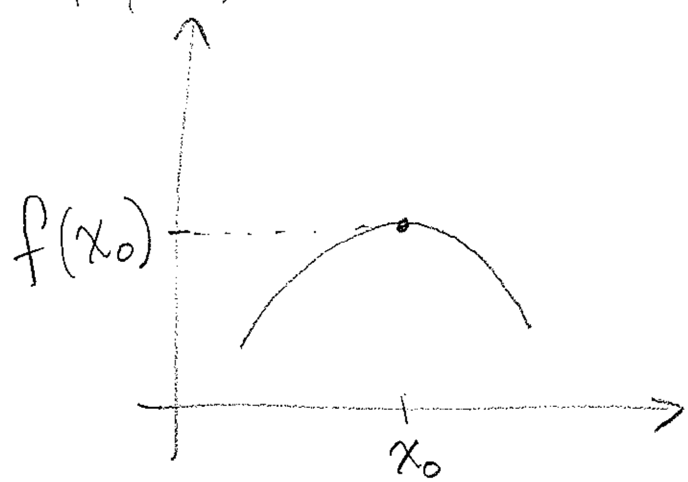
So near x_0 the graph of f looks like

$f''(x_0) > 0$



local min

$f''(x_0) < 0$



local max.

Taylor series for $f(x,y)$: For nice functions

$$f(x_0+h, y_0+k) = \underbrace{f(x_0,y_0) + f_x(x_0,y_0)h + f_y(x_0,y_0)k}_{\text{linear approx}}$$

next level of accuracy $\longrightarrow$ $+ah^2 + bhk + ck^2 + \underbrace{E(h,k)}_{\text{smaller than other terms.}}$

Q: What are a, b, c?

$$a = \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0,y_0) \quad b = \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) \quad c = \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(x_0,y_0)$$

Reason: Take $x_0 = y_0 = 0$, and $x = h$, $y = k$.

$$f(x,y) \approx \underbrace{f(\vec{0}) + f_x(\vec{0})x + f_y(\vec{0})y + ax^2 + bxy + cy^2}_{g(x,y)}$$

$f(\vec{0}) = g(\vec{0})$ ✓

$g_x = f_x(\vec{0}) + 2ax + by$

so $g_x(\vec{0}) = f_x(\vec{0})$ ✓

$g_{xx} = 2a$. So $f_{xx}(\vec{0}) = g_{xx}(\vec{0})$ gives the form for a.

Ex: Why Taylor series are so useful:

$$f(x,y) = \sin\left(\sqrt{1+\frac{x^2}{2+\cos y}} - e^{-xy}\right)$$

$\vec{0}$ is a crit. pt of this mess, but is it a max?

$$f(x,y) = \frac{1}{6}x^2 + xy + E(x,y)$$

This neither a min or a max:

$$f(x,x) = \frac{7}{6}x^2 + E(x,y) \quad \text{but}$$

$$f(x,-x) = -\frac{5}{6}x^2 + E(x,y)$$

So: Its a saddle.

2nd der test: Suppose (a,b) is a crit pt of f. Set

$$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}$$

If $D > 0$ <u>and</u> $f_{xx}(a,b) > 0$ then (a,b) is a local min

If $D > 0$ <u>and</u> $f_{xx}(a,b) < 0$ then (a,b) is a local max

If $D < 0$, then (a,b) is a saddle.

[If $D = 0$ or $f_{xx}(a,b) = 0$, break glass...]

<u>Ex:</u> $f = x^2 + y^2$ $\quad D = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0$

$f_{xx}(0,0) = 2 > 0$

$\boxed{\text{min}}$ ✓

<u>Ex:</u> $f = -x^2 - y^2$ $\quad D = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4 > 0$

$f_{xx}(0,0) = 2 < 0$

$\boxed{\text{max}}$ ✓

<u>Ex:</u> $f = x^2 - y^2$ $\quad D = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4 < 0$.

<u>Ex</u> $\underbrace{f = xy}$ $\quad D = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0$

also a saddle, from a worksheet.

Under the hood: Changing coordinates.