Lecture 9: Chain rule (Section 14.5)

Reminder: Exam Wed Sept 22.

Last time: Derivatives and linear approximation

$f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a,b) if

$$f(a+h, b+k) = \underbrace{f(a,b) + \frac{\partial f}{\partial x}(a,b)\, h + \frac{\partial f}{\partial y}(a,b)\, k}_{\text{the linear approximation to } f \text{ near } (a,b).} + E(h,k)$$

where

$$\lim_{(h,k)\to\vec{0}} \frac{E(h,k)}{\sqrt{h^2+k^2}} = 0$$

the linear approximation to f near (a,b).

Alternate notation: $\Delta x = h = x - a$ [Read in words]

$\Delta y = k = y - b$

$$f(x,y) - f(a,b) = \Delta f \approx f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$$

Approximately.

[You all know.]

$$\frac{d}{dt}\sin(t^2) = \cos(t^2)\cdot 2t$$

<u>Setup:</u> $f, g : \mathbb{R} \to \mathbb{R}$ then consider the composition $h = f \circ g$, that is $h(t) = f(g(t))$

<u>Chain rule:</u> $h'(t) = f'(g(t))\, g'(t)$

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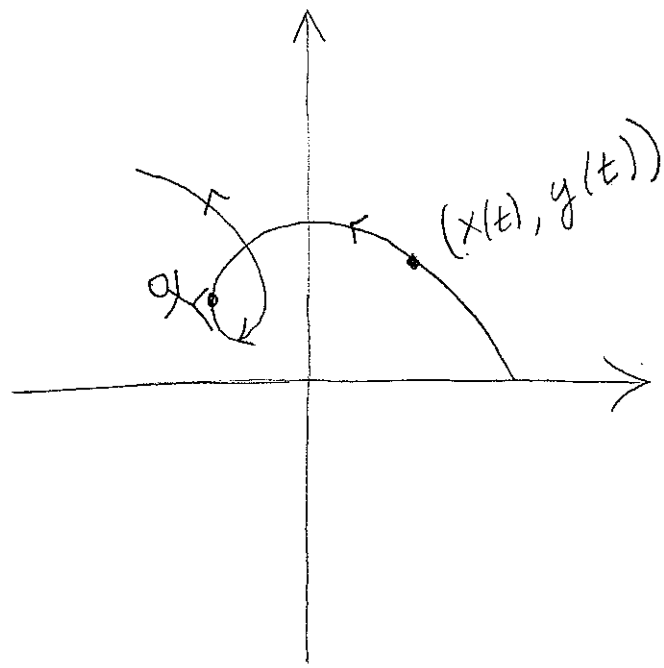
① $f : \mathbb{R}^2 \to \mathbb{R}$

② parameterized curve $(x(t), y(t))$



③ compose them

$$h(t) = f(x(t), y(t))$$

<u>Ex:</u> f = temperature as a fn of position
h = temperature observed as a fn of time.

<u>Q</u>: What is $h'(t)$?

[in terms of partials of f and der of x + y]

Want

$$h(t+\Delta t) = h(t) + h'(t)\,\Delta t + \underbrace{E(t)}_{\text{small comp to } \Delta t}$$

Know

$$x(t+\Delta t) = x(t) + x'(t)\,\Delta t + E_1(t)$$
$$y(t+\Delta t) = y(t) + y'(t)\,\Delta t + E_2(t)$$

and

$$f(x+\Delta x, y+\Delta y) \approx f(x,y) + f_x(x,y)\,\Delta x + f_y(x,y)\,\Delta y$$

So plug in and get

$$h(t+\Delta t) = f(x(t+\Delta t), y(t+\Delta t))$$

$$= f(\underbrace{x(t)}_{x} + \underbrace{x'(t)\Delta t + E_1(t)}_{\Delta x},\ y(t) + y'(t)\Delta t + E_2(t))$$

$$\approx f(x(t), y(t)) + f_x(x(t),y(t))\,(x'(t)\Delta t + E_1(t))$$
$$+ f_y(x(t), y(t))\,(y'(t)\Delta t + E_2(t))$$

$$\approx h(t) + \left(f_x(x(t),y(t))\, x'(t) + f_y(x(t),y(t))\, y'(t)\right)\Delta t$$

where I've thrown away the terms with $E_1(t)$, $E_2(t)$.

<u>Chain Rule:</u> $h(t) = f(x(t), y(t))$

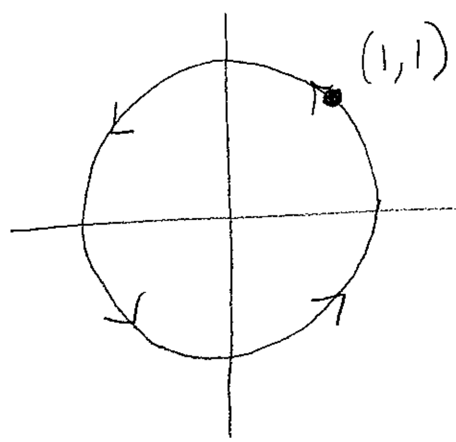
$$h'(t) = f_x(x(t),y(t))\,x'(t) + f_y(x(t),y(t))\,y'(t).$$

[Read in words.]

<u>Ex:</u> $f(x,y) = (x+y^2)^2$ $\quad$ $x(t) = \sqrt{2}\cos t$

$y(t) = \sqrt{2}\sin t$

$h(t) = f(x(t), y(t))$



<u>Find:</u> $h'(\pi/4)$

$x(\pi/4) = y(\pi/4) = 1$

$x'(t) = -\sqrt{2}\sin t$ $\qquad$ $x'(\pi/4) = -1$

$y'(t) = \sqrt{2}\cos t$ $\qquad$ $y'(\pi/4) = 1$

$$\frac{\partial f}{\partial x} = 2(x+y^2) \qquad \frac{\partial f}{\partial y} = 4y(x+y^2)$$

So

$$h'(\pi/4) = \frac{\partial f}{\partial x}(x(\pi/4), y(\pi/4)) \cdot x'(\pi/4)$$
$$+ \frac{\partial f}{\partial y}(x(\pi/4), y(\pi/4))\, y'(\pi/4)$$
$$= \frac{\partial f}{\partial x}(1,1)\,(-1) + \frac{\partial f}{\partial y}(1,1) \cdot 1$$
$$= 4 \cdot -1 + 8 \cdot 1 = 4.$$

Note

$$h(t) = (\sqrt{2}\ \cos t + 2\sin^2 t)^2$$

so can double check this directly.

Another way to think about the original chain rule. [Book likes.]

$f, g : \mathbb{R} \to \mathbb{R}$ and $h(t) = f(g(t))$

Set $y = f(x)$ and $x = g(t)$

so that y can be viewed as a fn of t.

Then

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Compare

$$\frac{dh}{dt}(t) = \frac{df}{dx}(x(t)) \cdot \frac{dg}{dt}(t)$$

Suppose $h(t) = f(x(t), y(t))$

Chain rule becomes

useful way to remember. "cancel dy with ∂y"

$$\frac{dh}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

or sometime write $z = f(x,y)$ and $x = x(t)$, $y = y(t)$

so that z is a fn of t. Then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$