Lecture 19: Integration along curves (16.2)

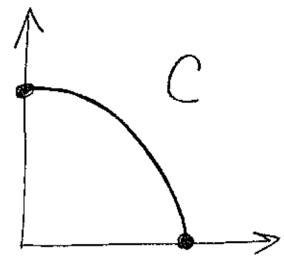
<u>Last time:</u>

C a curve in $\mathbb{R}^2$

$f: \mathbb{R}^2 \to \mathbb{R}$

$$\int_C f \, ds = \int_a^b f(\vec{r}(t)) \, |\vec{r}\,'(t)| \, dt$$

where $\vec{r}: [a,b] \to \mathbb{R}^2$ is a parameterization of C.



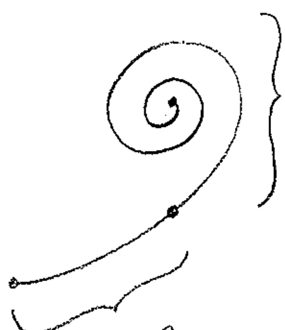
$f(x,y) = 10x^2$

$$\int_C f \, ds = \tfrac{5}{2}\pi$$

[Focus today: meaning of such line integrals.]

Q: What is the average value of f on C?

<u>Ex:</u>



$f = 3$
Length $= 5$

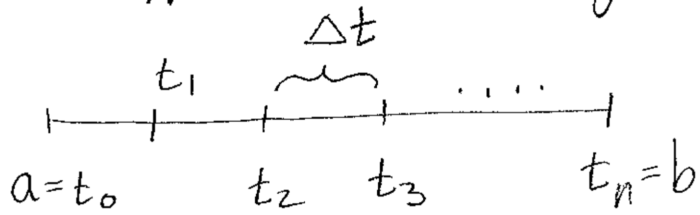
$f = 2$
Length $= 3$

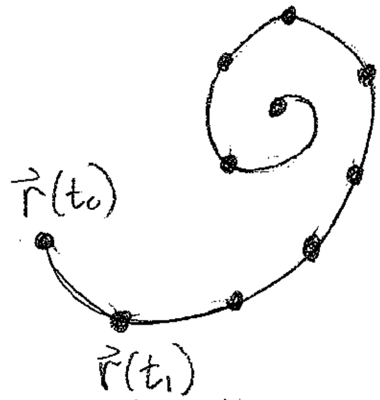
$$\text{Average} = \begin{pmatrix}\text{portion} \\ \text{of } C \\ \text{where} \\ f=3\end{pmatrix} \cdot 3 + \begin{pmatrix}\text{portion} \\ \text{of } C \\ \text{where} \\ f=2\end{pmatrix} \cdot 2$$

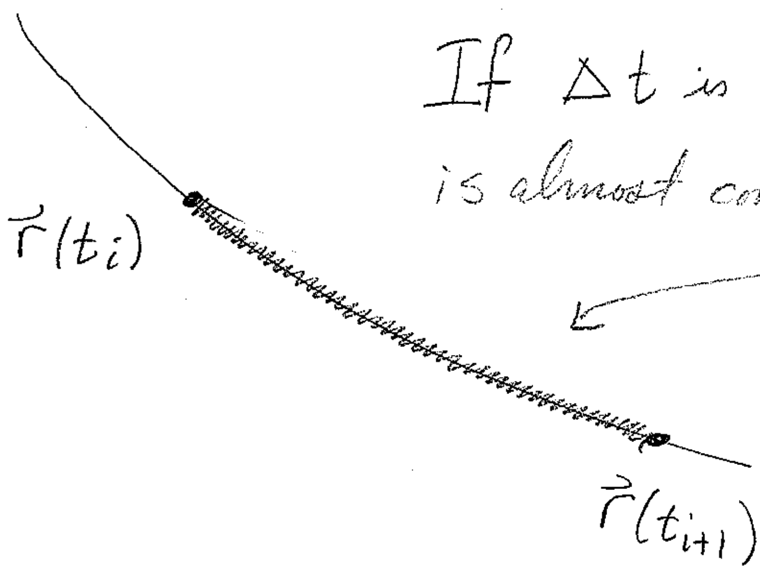
$$= \frac{5}{8} \cdot 3 + \frac{3}{8} \cdot 2 = \frac{21}{8}$$

$$= 2\tfrac{5}{8}$$

To approx. the average







If Δt is small then f is is almost const on with value $f(t_i)$.

So this segment contributes $\approx \dfrac{(\text{length of segment})}{(\text{length of } C)} \cdot f(t_i)$ to the average.

Last time, saw

$$(\text{length of seg}) \approx |\vec{r}(t_{i+1}) - \vec{r}(t_i)| \approx |\vec{r}\,'(t_i)|\,\Delta t$$

So

$$\text{Average} \approx \sum_{i=0}^{n-1} \frac{(\text{len of } i^{th} \text{ seg})}{(\text{len of } C)} f(t_i)$$

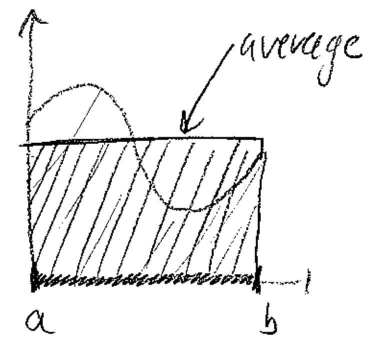
$$\approx \frac{1}{\text{len of } C} \sum_{i=0}^{n-1} f(t_i)\,|\vec{r}\,'(t_i)|\,\Delta t$$

As $\Delta t \to 0$, get

$$\text{Average} = \frac{1}{\text{Length } C} \int_a^b f(t)\,|\vec{r}\,'(t)|\,dt$$

$$= \frac{1}{\text{Length } C} \int_C f\,ds.$$

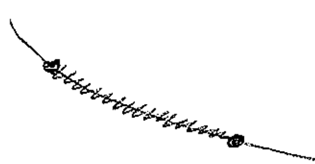
Compare: $\begin{array}{l}\text{Average of}\\ f \text{ on } a,b\end{array} = \dfrac{1}{b-a}\displaystyle\int_a^b f(t)\,dt.$



Ex: Average of $f(x) = 10x^2$ on

$$= \frac{1}{\text{Length } C} \int_C f\,ds = \frac{1}{\pi/2} \cdot \frac{5\pi}{2} = 5.$$

$ds = |r'(t)|\,dt \longrightarrow |\vec{r}\,'(t_i)|\,\Delta t$ in Riemann sum.



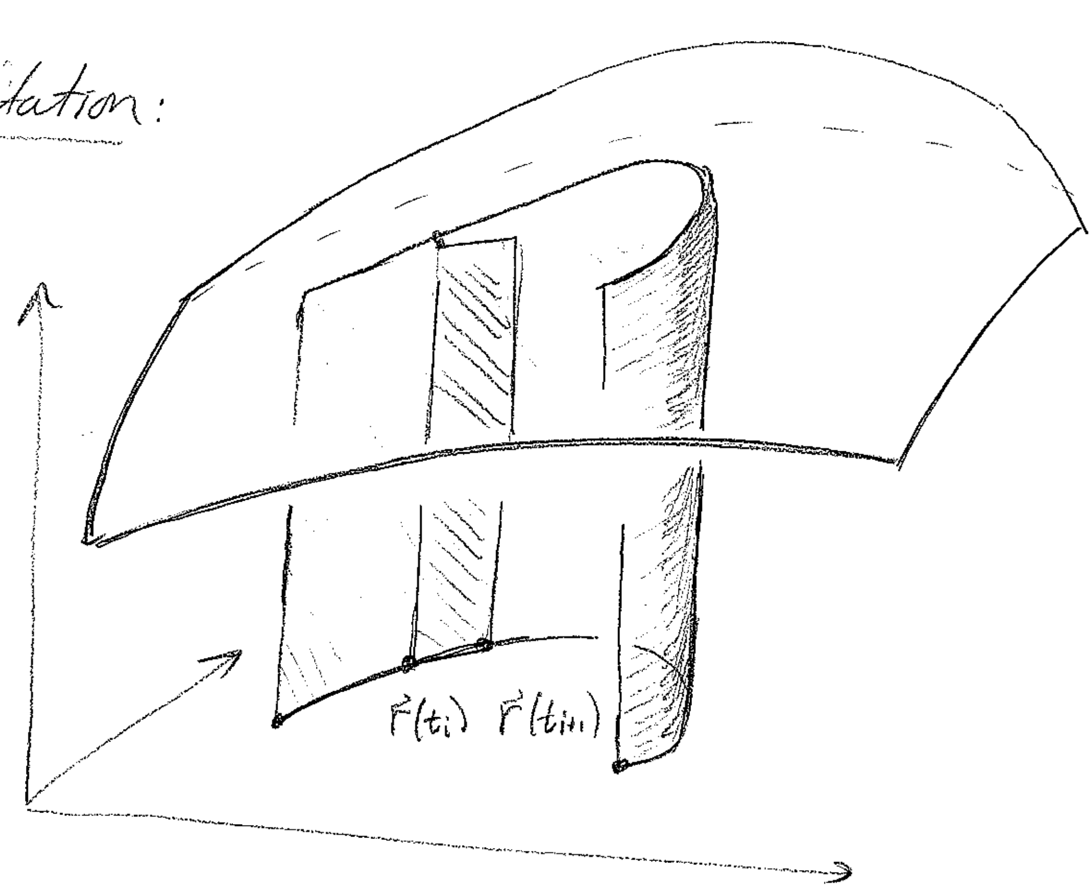
called the "arc length" element.

Note: $\displaystyle\int_C 1\,ds = \text{Length of } C.$

Another interpretation:

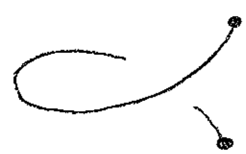
$\int_C f\,ds =$

Area above C below the graph of f.



Same formula let's define $\int_C f\,ds$ for a curve C in $\mathbb{R}^3$.

Ex: $C =$ one turn of helix



Made of material with density

$$\rho(x,y,z) = x^2 + z^2 \ \text{g/cm}$$

What is the total mass of C? $\int_C \rho\,ds$

Take $\vec{r}:[0,2\pi] \longrightarrow \mathbb{R}^3$ $\quad \vec{r}(t) = (\cos t, \sin t, t)$

$\vec{r}'(t) = (-\sin t, \cos t, 1)$

$|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$

$$\int_C \rho \, ds = \int_0^{2\pi} \rho(\vec{r}(t)) \, |\vec{r}'(t)| \, dt$$

$$= \int_0^{2\pi} \rho(\cos t, \sin t, t) \sqrt{2} \, dt$$

$$= \sqrt{2} \int_0^{2\pi} \cos^2 t + t^2 \, dt = \sqrt{2} \int_0^{2\pi} \cos^2 t \, dt + \int_0^{2\pi} t^2 \, dt$$

$$\sqrt{2}\pi + \frac{\sqrt{2}(2\pi)^3}{3} \approx 121.37 \text{ g}$$

<u>Notes:</u> Unit check:

$$\underset{g/cm}{\rho(\vec{r}(t))} \; \underset{cm/s}{|\vec{r}'(t)|} \; \underset{s}{dt} = g$$

Note on $\int_0^{2\pi} \cos^2 t \, dt$

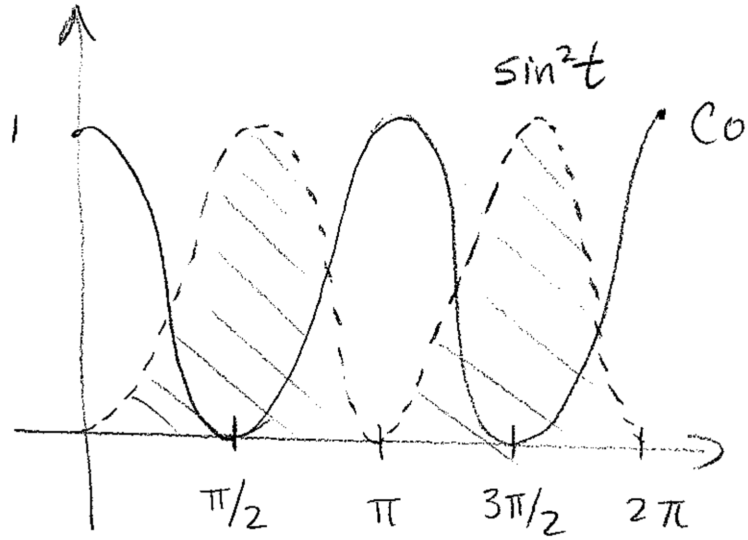
(A) Trig identities: $\cos 2t = \cos^2 t - \sin^2 t$
$= 2\cos^2 t - 1$

$\Rightarrow \quad \cos^2 t = \frac{1}{2}(1 + \cos 2t)$

$$\int_0^{2\pi} \cos^2 t \, dt = \frac{1}{2}\int_0^{2\pi} (1 + \cos 2t)\, dt$$

$$= \frac{1}{2}\left(t - \frac{1}{2}\sin 2t\right)\Bigg|_{t=0}^{2\pi} = \pi$$

(B)



Notice $\int_0^{2\pi} \sin^2 t \, dt = \int_0^{2\pi} \cos^2 t \, dt$

Since $2\pi = \int_0^{2\pi} 1 \, dt = \int_0^{2\pi} \sin^2 t \, dt + \int_0^{2\pi} \cos^2 t \, dt$

we get $\int_0^{2\pi} \sin^2 t \, dt = \pi$.