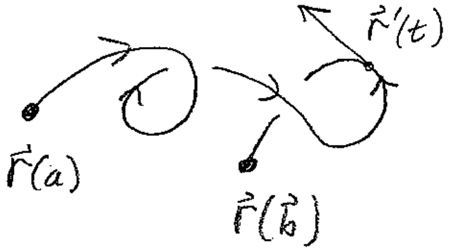
Lecture 18: Integration along curves (13.3 and 16.2)

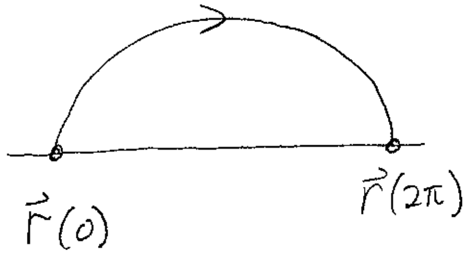
Last time: $\vec{r} : \mathbb{R} \to \mathbb{R}^2$ or $\mathbb{R}^3$

Length: $\vec{r} : [a,b] \to \mathbb{R}^3$



$$\text{Length} = \int_a^b |\vec{r}(t)|\, dt$$

Ex: Cycloid $\quad \vec{r}(t) = (t - \sin t,\ 1 - \cos t) \leftarrow$ Copied wrong last time.



$$\vec{r}\,'(t) = (1 - \cos t,\ \sin t)$$

$$|\vec{r}\,'(t)| = \sqrt{\underbrace{(1-\cos t)^2}_{1 - 2\cos t + \cos^2 t} + \sin^2 t}$$

$$= \sqrt{2 - 2\cos t} = \sqrt{4 \sin^2 t/2} = 2 \sin t/2$$

since $\sin t/2 \geq 0$ for t in $[0, 2\pi]$

Since

$$\cos 2s = \cos^2 s - \sin^2 s$$
$$= 1 - 2\sin^2 s$$

$$\Rightarrow 2\sin^2 s = 1 - \cos 2s$$

Thus

$$\text{Length} = \int_0^{2\pi} |\vec{r}\,'(t)|\, dt = \int_0^{2\pi} 2 \sin t/2\, dt = -4\cos t/2 \Big|_{t=0}^{2\pi}$$

$$= 4 - (-4) = 8.$$

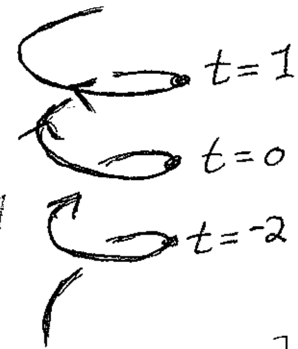
Really two [closely related] concepts:

<u>Curve</u>: A set of points in $\mathbb{R}^3$ looking like:

<u>Parameterization</u>:

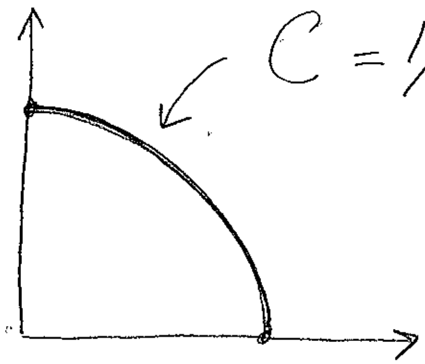
$\vec{r}: \mathbb{R} \longrightarrow \mathbb{R}^3$

<u>Instructions</u> for <u>moving</u> <u>along</u> a curve



[Any curve has <u>many</u> parametrizations]

<u>Ex</u>:



$C = 1/4$ of unit circle in $\mathbb{R}^2$

ⓐ $\vec{r}: [0, \pi/2] \longrightarrow \mathbb{R}^2 \qquad \vec{r}(t) = (\cos t, \sin t)$

$\vec{r}\,'(t) = (-\sin t, \cos t)$

$$\text{Length} = \int_0^{\pi/2} |\vec{r}\,'(t)|\, dt \qquad |\vec{r}\,'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t} = 1$$

$$= \int_0^{\pi/2} 1\, dt = \boxed{\pi/2}$$

ⓑ $\vec{r}: [0, 1] \longrightarrow \mathbb{R}^2 \qquad \vec{r}(t) = (t, \sqrt{1-t^2})$

$$\vec{r}\,'(t) = \left(1, \frac{-t}{\sqrt{1-t^2}}\right)$$

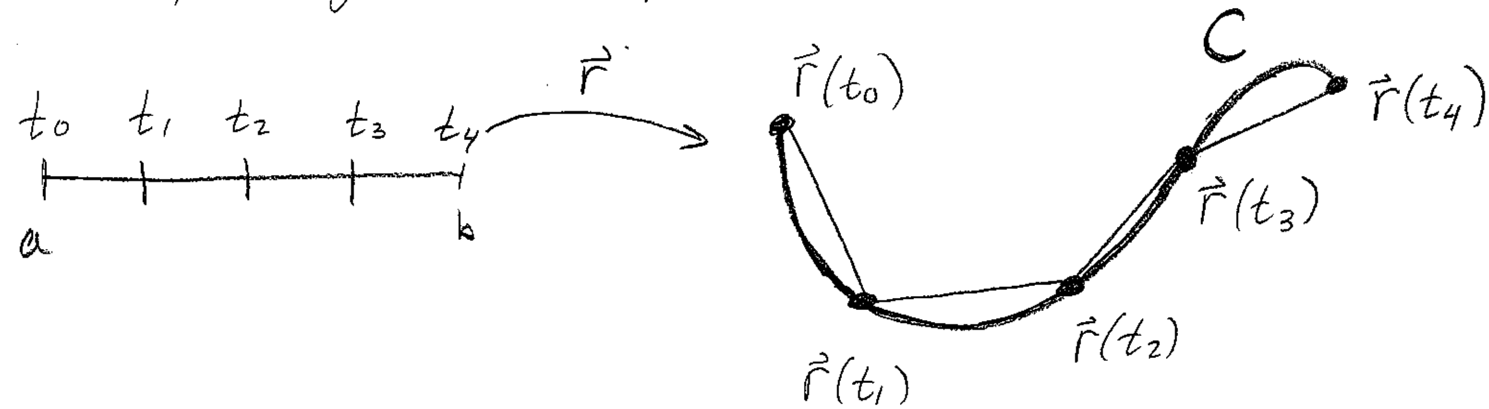
$$\vec{r}\,'(t) = \sqrt{1 + \frac{t^2}{1-t^2}} = \frac{1}{\sqrt{1-t^2}}$$

$$\text{Length} = \int_0^1 \frac{1}{\sqrt{1-t^2}}\, dt$$

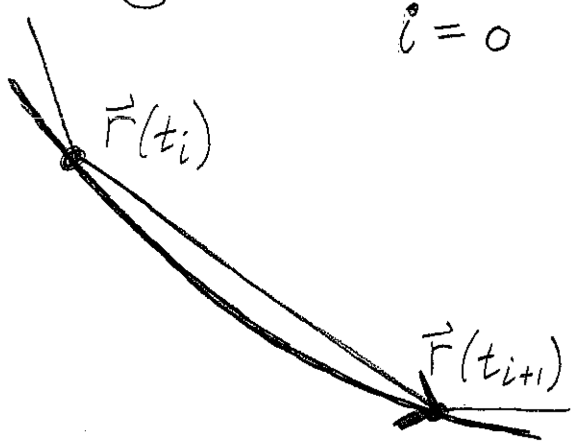
$$= \arcsin(t) \Big|_0^1 = \pi/2 - 0 = \boxed{\pi/2}$$

[Notice that the length only depended on the curve, not the parameterization.]

Another point of view: Approximate by straight segments:



$$\text{Length of } C \approx \sum_{i=0}^{n} |\vec{r}(t_{i+1}) - \vec{r}(t_i)|$$



Linear approximation:

$$\vec{r}(t_i + \Delta t) \approx \vec{r}(t_i) + \vec{r}\,'(t_i)\Delta t$$

Take $\Delta t = t_{i+1} - t_i$ so get

$$\vec{r}(t_{i+1}) - \vec{r}(t_i) \approx \vec{r}\,'(t_i)\Delta t.$$ Thus

$$\text{Length of } C \approx \sum_{i=0}^{n} |\vec{r}(t_{i+1}) - \vec{r}(t_i)| \approx \sum_{i=0}^{n} \vec{r}\,'(t_i)\Delta t$$

As $\Delta t \to 0$, the Riemann sums at right converge to $\int_a^b |\vec{r}'(t)|\, dt$. As the other approx get better as Δt, we've found again that

$$\text{Length of } C = \int_a^b |\vec{r}'(t)|\, dt$$

<u>Integration along a curve:</u> (Line integrals 16.2)

C curve in $\mathbb{R}^2$

$f: \mathbb{R}^2 \to \mathbb{R}$ a function

$$\int_C f\, ds = \int_a^b f(r(t))\, |r'(t)|\, dt$$

where $\vec{r}: [a,b] \longrightarrow \mathbb{R}^2$ is a parameterization of C.

[Turns out <u>not</u> to depend on $\vec{r}$, just C]

Some meanings:

① f = temperature

$$\text{Average temp along } C = \frac{1}{\text{Length } C} \int_C f \, ds$$

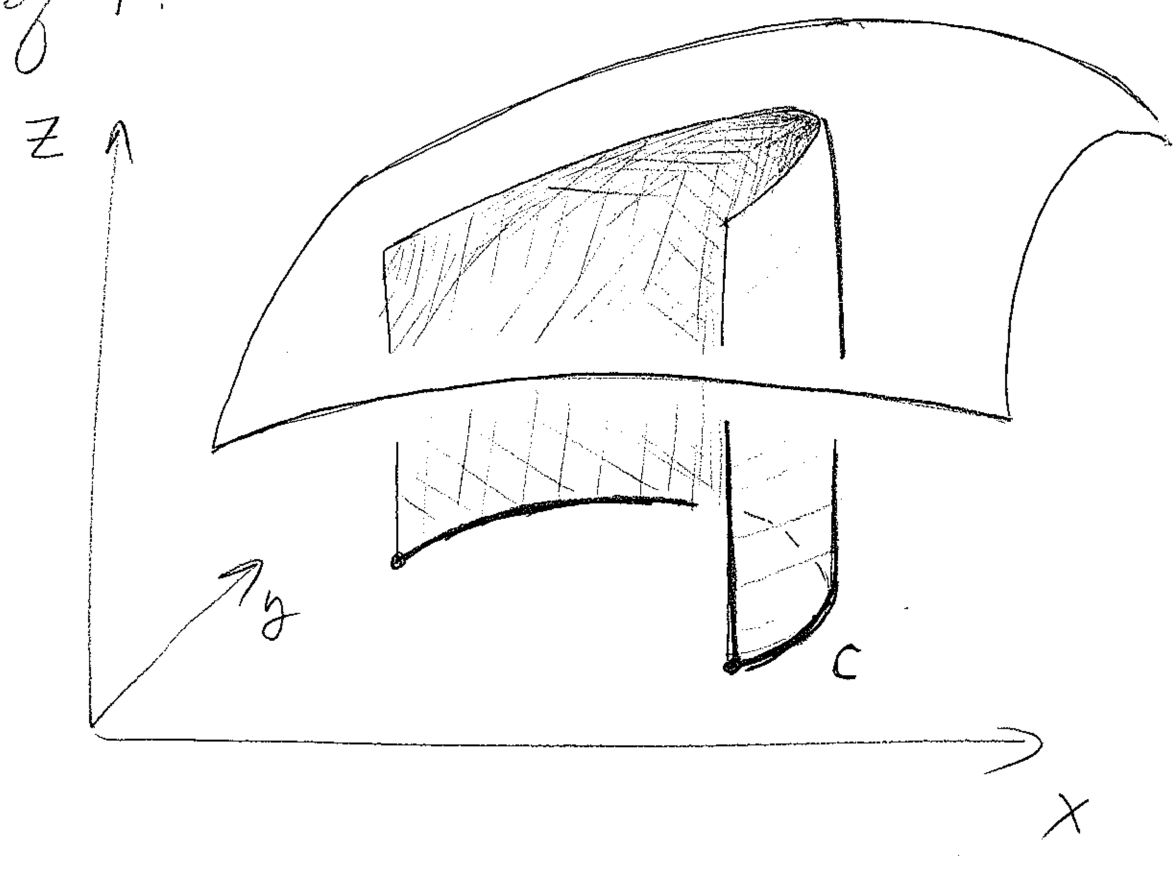
Compare:
Average of $f: \mathbb{R} \to \mathbb{R}$ on $[a,b]$ is

$$\frac{1}{b-a} \int_a^b f(x) \, dx$$

② f = density of curve (mass/unit length)

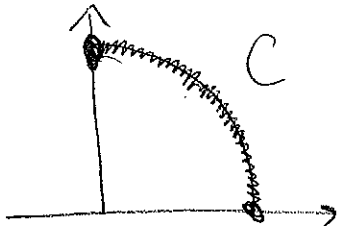
$$\text{mass of curve} = \int_C f \, ds$$

③ Area of region above C and below the graph of f.



Ex: Find the average of $f(x,y) = 10x^2$

on C



$\vec{r}(t) = (\cos t, \sin t) \quad 0 \le t \le \pi/2$

$$\int_C f\, ds = \int_0^{\pi/2} f(\vec{r}(t))\, |\vec{r}\,'(t)|\, dt$$

$$= \int_0^{\pi/2} 10\cos^2(t) \cdot 1\, dt = 10 \int_0^{\pi/2} \cos^2 t\, dt$$

$$= \frac{5}{2}\pi$$

Note: $\cos^2 t = \frac{1}{2}(1 + \cos 2t)$, so

$$\int_0^{\pi/2} \cos^2 t\, dt = \frac{1}{2}\int_0^{\pi/2} (1 + \cos 2t)\, dt$$

$$= \frac{1}{2}\left(t + \frac{1}{2}\sin 2t\right)\Big|_{t=0}^{t=\pi/2} = \frac{\pi}{4}$$

$$\text{Average} = \frac{1}{\text{Length}} \int_C f\, ds = \frac{\frac{5\pi}{2}}{\pi/2} = 5$$