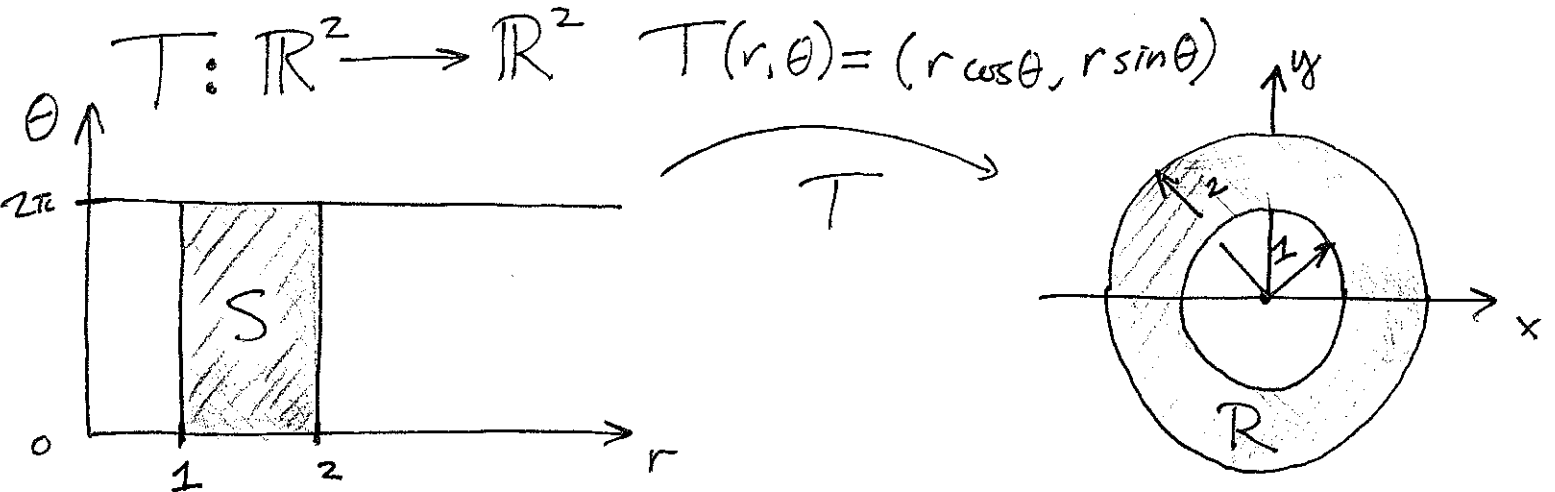
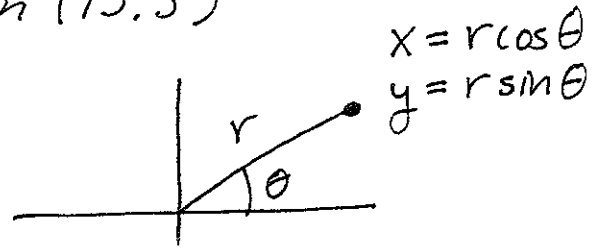


# Lecture 27: Polar Coordinates (15.4) and Applications of Integration (15.5)

Last time: Polar coordinates



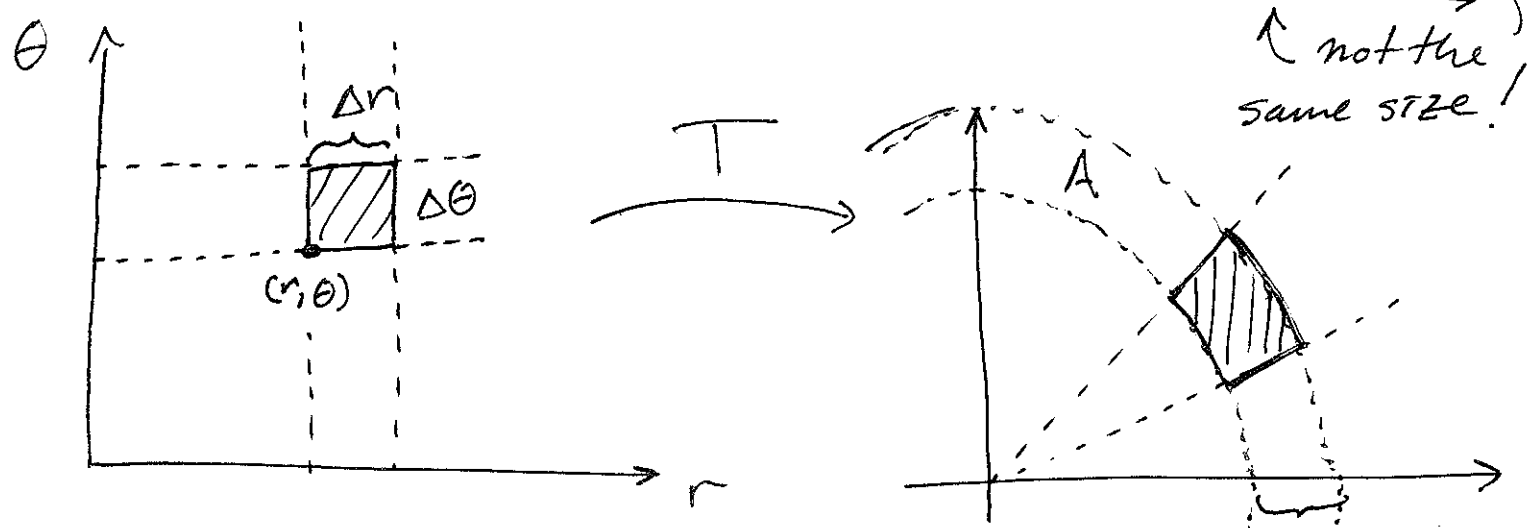
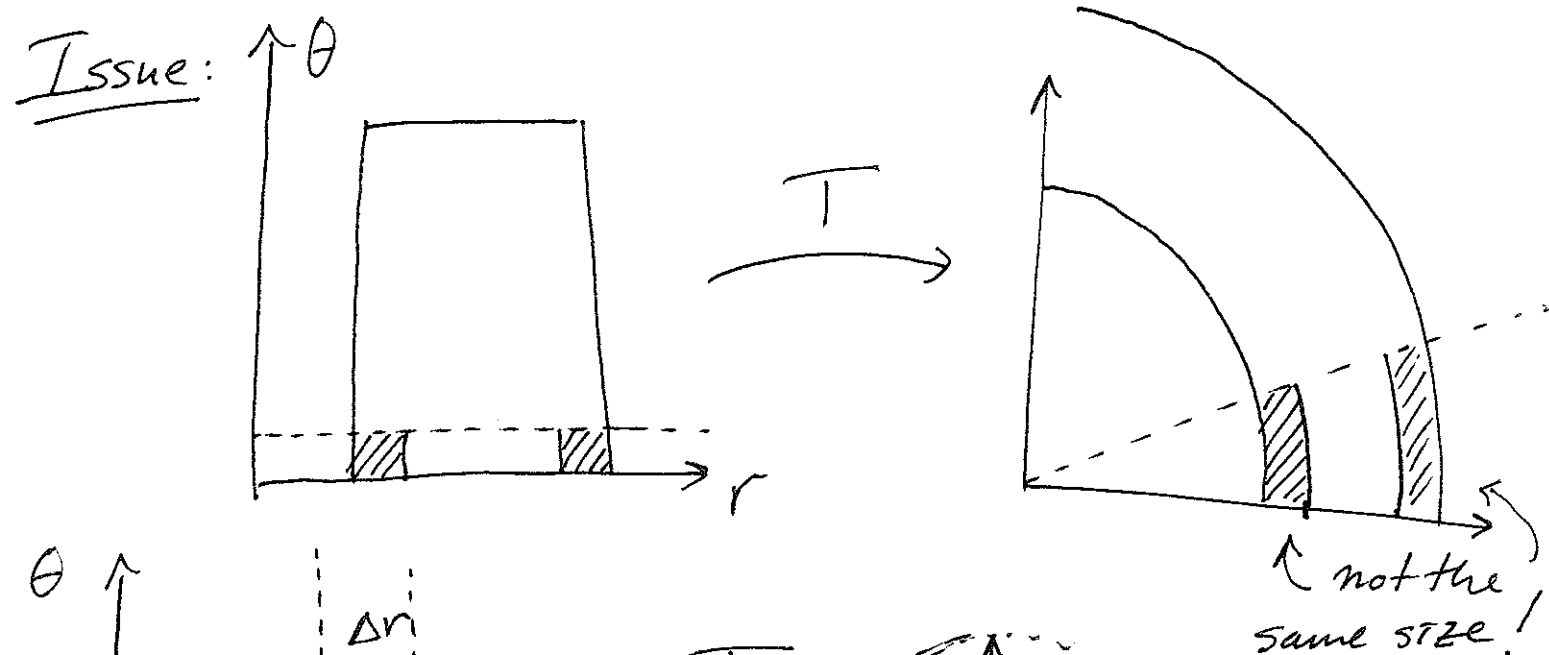
$R = T(S)$

Goal: Relate  $\iint_R f \, dA$  to an integral over  $S$ .

First try:  $\iint_R 1 \, dA \stackrel{\text{Guess!}}{=} \iint_S 1 \, dA = \int_0^{2\pi} \int_1^2 1 \, dr \, d\theta$   
 $= \int_0^{2\pi} r \Big|_{r=1}^2 \, d\theta = \int_0^{2\pi} 1 \, d\theta = 2\pi.$

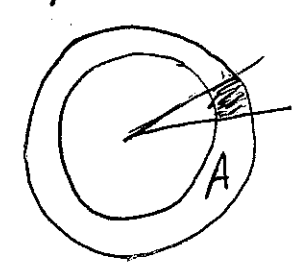
Is this right?

$\iint_R 1 \, dA = \text{Area}(R) = \text{Area}(\text{disc of rad } 2) - \text{Area}(\text{disc of rad } 1)$   
 $= \pi \cdot 2^2 - \pi \cdot 1^2 = 3\pi \quad \text{So } \underline{\underline{NO.}}$



$$\text{Area}(T(\square)) = \left( \begin{array}{l} \text{portion } T(\square) \\ \text{is of the annulus } A \end{array} \right) \cdot \left( \begin{array}{l} \text{Area of} \\ A \end{array} \right)$$

$$= \frac{\Delta\theta}{2\pi} \left( \underbrace{\pi(r+\Delta r)^2 - \pi r^2}_{\pi(r^2 + 2r\Delta r + \Delta r^2 - r^2)} \right)$$



$$= r \Delta r \Delta\theta + \frac{1}{2} \Delta r^2 \Delta\theta \approx r \Delta r \Delta\theta \text{ if } \Delta r \text{ is small.}$$

$$\text{Area}(R) = \sum_{\text{subrects of } S} \text{Area}(T(\square)) \approx \sum r \Delta r \Delta\theta \approx \iint_S r \, dA$$

So

$$\begin{aligned}
 \text{Area}(R) &= \iint_R 1 \, dA = \iint_S r \, dA = \int_0^{2\pi} \int_0^1 r \, dr \, d\theta \\
 &= \int_0^{2\pi} \left. \frac{r^2}{2} \right|_{r=1}^2 d\theta = \int_0^{2\pi} \frac{2^2}{2} - \frac{1^2}{2} d\theta \\
 &= \int_0^{2\pi} \frac{3}{2} d\theta = 3\pi
 \end{aligned}$$

which matches our geometric answer!

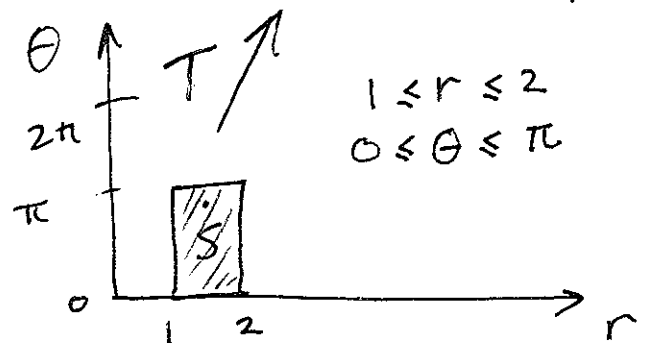
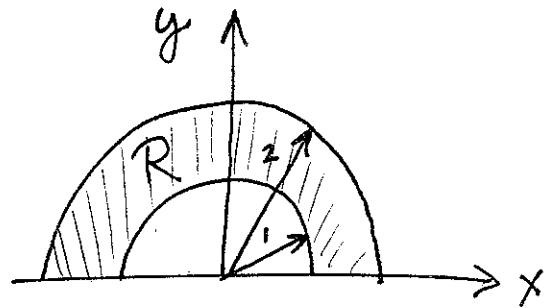
Summary: In polar coordinates,  $dA = r \, dr \, d\theta$   
not  $dr \, d\theta$ .

Ex:  $\iint_R f(x,y) \, dA$  for  $f(x,y) = y$ .

$$= \iint_S f(T(r,\theta)) \, r \, dr \, d\theta$$

$$= \int_0^{\pi} \int_1^2 (r \sin \theta) \, r \, dr \, d\theta$$

$$= \int_0^{\pi} \left. \frac{r^3}{3} \sin \theta \right|_{r=1}^{r=2} d\theta$$

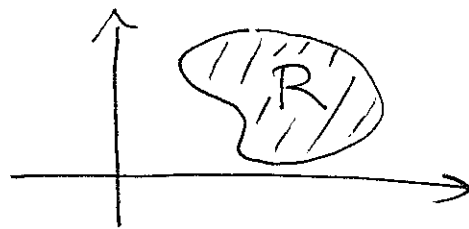


$$\int_0^{\pi} \frac{1}{3} \sin \theta (2^3 - 1^3) d\theta = \frac{7}{3} \int_0^{\pi} \sin \theta d\theta$$

$$= \frac{7}{3} (-\cos \theta) \Big|_{\theta=0}^{\pi} = \frac{14}{3}$$

Challenge: Get the same answer w/o polar coord!

Applications: (Section 15.5).



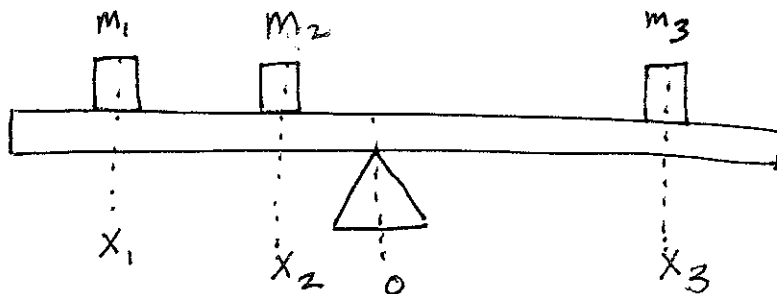
$$\text{Area} = \iint_R 1 dA$$

$$\text{Mass} = \iint_R \rho(x,y) dA \quad \text{where } \rho(x,y) = \frac{\text{mass}}{\text{area}}$$

$$\text{Average} = \frac{1}{\text{Area}(R)} \iint f dA$$

[ Already mentioned these, are just like line int case. As are the next ones, but we never actually talked about these in detail. Let's do that now... ]

Center of mass:

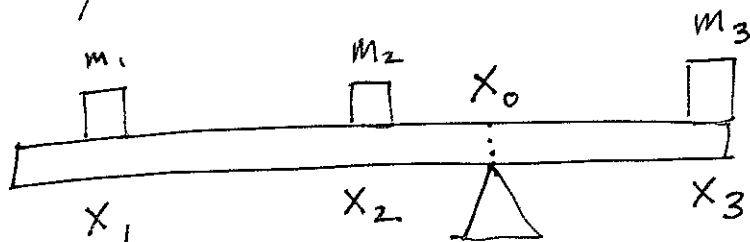


Balanced means:

$$m_1 x_1 + m_2 x_2 + m_3 x_3 = 0$$

Suppose we want to find the balance point, i.e. solve

for  $x_0$



where

$$m_1(x_1 - x_0) + m_2(x_2 - x_0) + m_3(x_3 - x_0) = 0$$

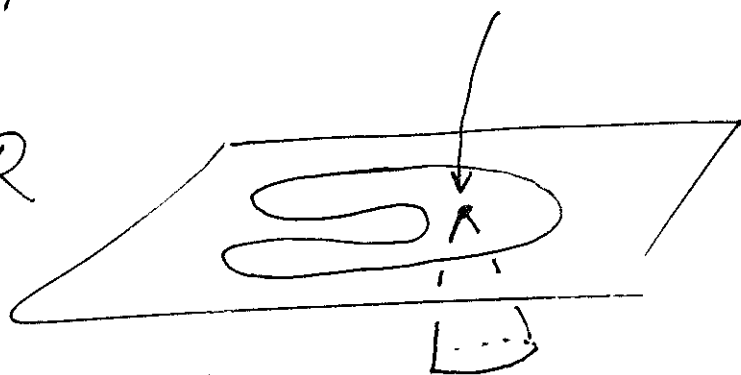
$$\Rightarrow m_1 x_1 + m_2 x_2 + m_3 x_3 = (m_1 + m_2 + m_3) x_0$$

$$\Rightarrow x_0 = \frac{1}{\text{total mass}} (m_1 x_1 + m_2 x_2 + m_3 x_3) \quad \text{"Center of mass"}$$

Note: If the  $m_i$  are all equal, then

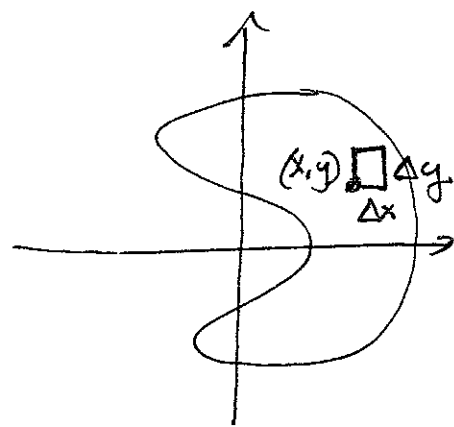
$x_0$  is just the average of the  $x_i$  Center of mass.

Now consider a region  $R$   
in  $\mathbb{R}^2$  with density  
given by  $\rho(x, y)$



$$M_x = \frac{1}{\text{Mass}} \sum_{\text{rect making up } R} X \cdot \begin{matrix} \text{mass of} \\ \text{rect} \end{matrix}$$

$$= \frac{1}{\text{Mass}} \sum_{\text{boxes}} X (\rho(x,y) \Delta x \Delta y)$$



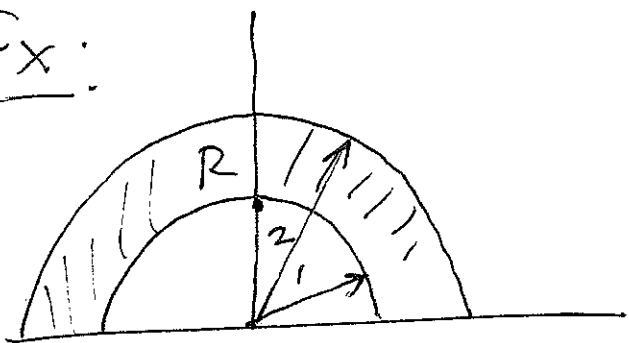
$$\approx \frac{1}{\text{Mass}} \iint_R x \rho(x,y) dA = M_x$$

Note: If  $\rho$  is constant =  $\rho_0$ , then

$$M_x = \frac{1}{\iint_R \rho_0 dA} \iint_R x \rho_0 dA = \frac{1}{\rho_0} \frac{1}{\iint_R dA} \rho_0 \iint_R x dA$$

$$= \frac{1}{\text{Area}(R)} \iint_R x dA = \text{Average of } x \text{ coord on } R.$$

Ex:



Density = const.

$m_x = 0$  by symmetry.

$$m_y = \text{Average of } y = \frac{1}{\text{Area}} \iint_R y dA$$

$$= \frac{1}{3\pi/2} \cdot \left(\frac{14}{3}\right) = \frac{28}{9\pi}$$

CANT QUITE BALANCE IT!

$$\approx 0.9902$$