Lecture 22: Conservative Vector Fields (16.3)

Previously on Math 241:

Reminder: Exam next Wed (Oct 20)
See http://dunfield.info/241/

Fund. Thm. of Line Integrals: $f: \mathbb{R}^n \to \mathbb{R}$ differentiable

C a curve joining A to B.

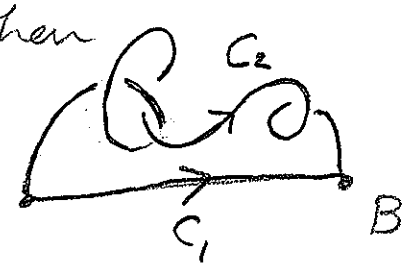
$$\int_C \nabla f \cdot d\vec{r} = f(B) - f(A)$$

A vector field $\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$ is conservative if $\vec{F} = \nabla f$ for some $f: \mathbb{R}^n \to \mathbb{R}$

By the Fund Thm, if $\vec{F}$ is conservative then

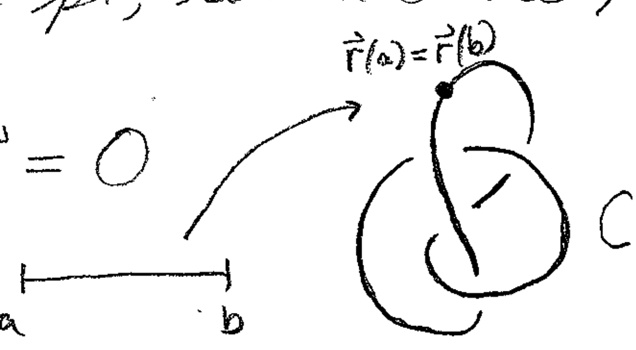
(a) Independence of path: If C_1 and C_2 are two paths joining A to B, then

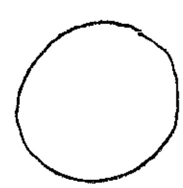
$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$



(b) If C is a closed curve (i.e. starts and ends at the same pt, like a circle) then

$$\int_C \vec{F} \cdot d\vec{r} = 0$$





Reason: If $\vec{r}(a) = \vec{r}(b)$, then

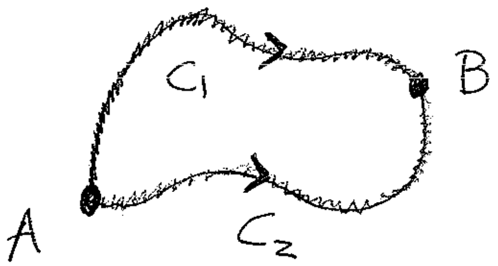
[Skip ahead to the examples.]

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = 0$$

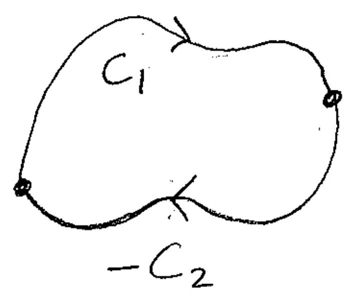
same input

Note: Actually these two conditions are equivalent.

For example if (b) is true for $\vec{F}$ consider



Let $-C_2$ denote C_2 backwards

Take $C =$ 

Then

$$0 = \int_C \vec{F} \cdot dr = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{-C_2} \vec{F} \cdot d\vec{r}$$

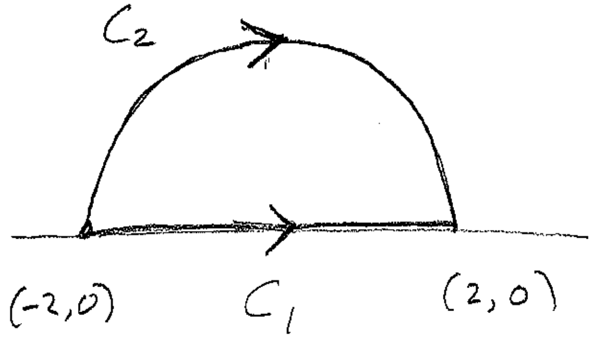
$$= \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot dr$$

$$\Rightarrow \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}.$$

[Conversely, if (a) holds for $\vec{F}$, then break a closed curve into 2 segments to see that $\int_C \vec{F} \cdot ds = 0$]

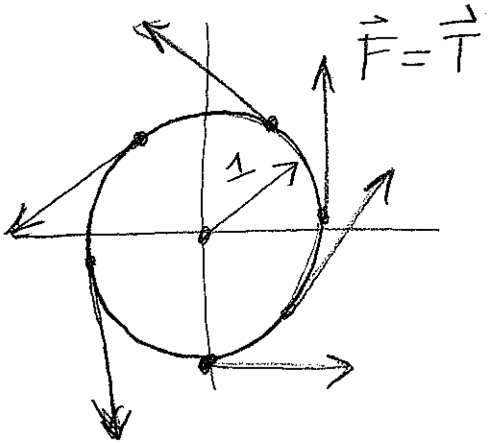
Examples of nonconservative vector fields:

① $\vec{F} = (y, 0)$ [from yesterday's worksheet]



$$\left.\begin{aligned} \int_{C_1} \vec{F} \cdot d\vec{r} &= 0 \\ \int_{C_2} \vec{F} \cdot d\vec{r} &= 2\pi \end{aligned}\right\} \text{ different!}$$

② $\vec{F} = (-y, x)$ [from 1^{st} lecture on vector fields.]



$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T}\, ds$$

$$= \int_C ds = 2\pi \neq 0$$

__Note__: Being conservative is kinda subtle. For example, is $\vec{F} = (y, x)$ conservative? [Compare ② above]

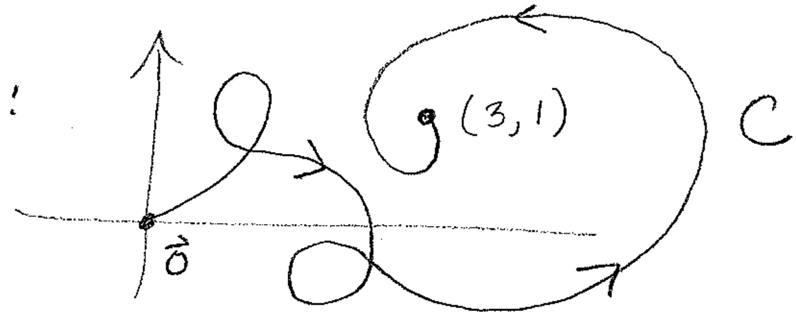
__A__. Yes. Need $f(x,y)$ with $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = x$

First cond says $f = \int y\, dx = xy + C(y)$

$\uparrow$ treated as a constant

Second is similar, so take $f = xy$.

Usefullness is clear:



$$\int_C \vec{F} \cdot ds = f(3,1) - f(0,0) = 3.$$

[Will give two tests for $\vec{F}$ to be conservative, but first need to introduce some terms...]

Focus on a vector field $\vec{F}$ on $\mathbb{R}^2$, with domain D.

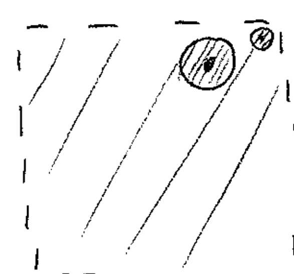
Ex: $\vec{F} = (-y, x) \quad D = \mathbb{R}^2$

$$\vec{F} = \frac{1}{x^2+y^2}(-y, x) \qquad D = \{(x,y) \neq (0,0)\}$$

Properties D may have:

Open: Roughly, D contains none of its boundary points. Precisely, each pt of D is the center of a disc also contained in D.

Ex: $D = \left\{ (x,y) \,\middle|\, \begin{array}{l} 0 < x < 1 \\ 0 < y < 1 \end{array} \right\}$

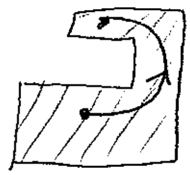
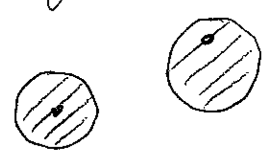


$D = \mathbb{R}^2$

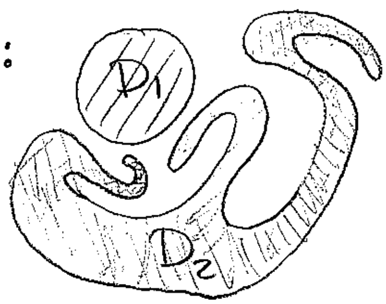
$D = \{(x,y) \neq (0,0)\}$

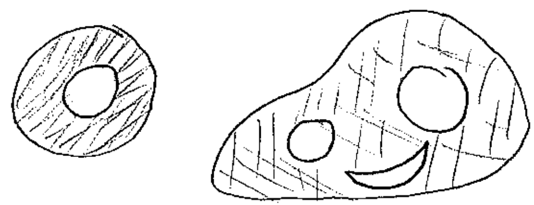
Closed: Opposite of open.
[D is closed if and only if its complement is open.]

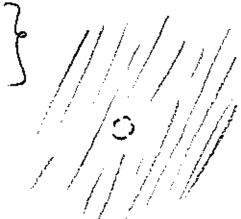
Connected: Any two points in D can be joined by a path inside D.

Ex:  Non Ex: 

Simply Connected: Connected + no holes.

Ex: 

Non Ex: 

$\{(x,y) \neq (0,0)\}$ 

Thm A: Suppose $\vec{F}$ is a vector field on an open connected set D. Then $\vec{F}$ is conservative if and only if $\int_C F \cdot d\vec{r}$ is path independent.

Thm B: Suppose $\vec{F} = (P, Q)$ is a vector field on an open simply connected set D. Then $\vec{F}$ is conservative if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on D.

[Thm A is nice, but hard to apply. Will talk about next time.]

<u>Reason for Thm B:</u> Suppose $\vec{F} = \nabla f = \left(\underset{\overset{\shortparallel}{P}}{\frac{\partial f}{\partial x}}, \underset{\overset{\shortparallel}{Q}}{\frac{\partial f}{\partial y}} \right)$

Then

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial y \partial x}$$

and

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} \frac{\partial f}{\partial y} = \frac{\partial^2 f}{\partial x \partial y}$$ and mixed partials are equal!

[We'll come back to why $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Rightarrow \vec{F}$ conserv.
just before Thanksgiving...]

<u>Ex:</u> $\vec{F} = (y, x)$ and $\frac{\partial P}{\partial y} = 1 = \frac{\partial Q}{\partial x} \Rightarrow$ conserv.

$\vec{F} = (-y, x)$ and $\frac{\partial P}{\partial y} = -1 \neq 1 = \frac{\partial Q}{\partial x} \Rightarrow$ not conserv.

<u>Ex:</u> $\vec{F} = \frac{1}{x^2+y^2}(-y, x)$ on $D = \{(x,y) \neq 0\}$

On HW: Check that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ <u>but</u> not path indep, hence not conservative.

<u>Point:</u> D is not simply conn.