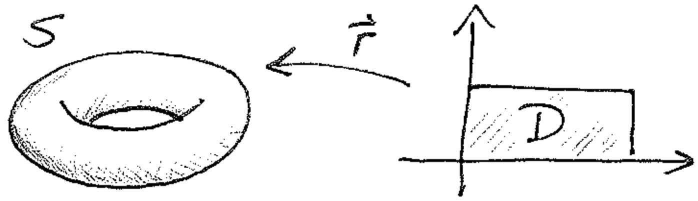
Lecture 35: More on surface integrals (16.7)

<u>Last time:</u> S surface in $\mathbb{R}^3$

$f: \mathbb{R}^3 \to \mathbb{R}$

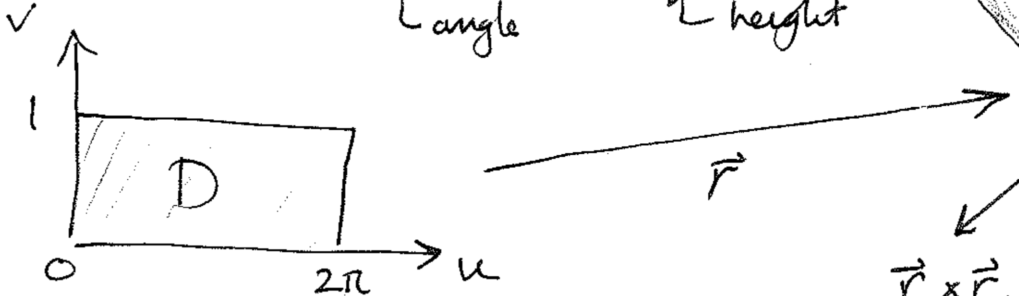


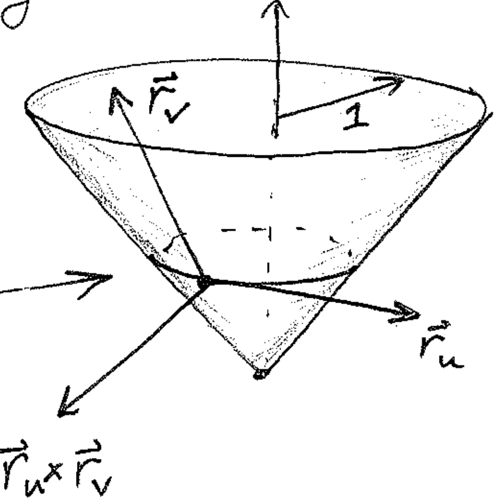
$$\iint_S f\, dA = \iint_D f(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

<u>Ex:</u> Find the average of $f(x,y,z) = xy + z$ on S:

$\vec{r}(u,v) = (v\cos u, v \sin u, v)$

angle — height





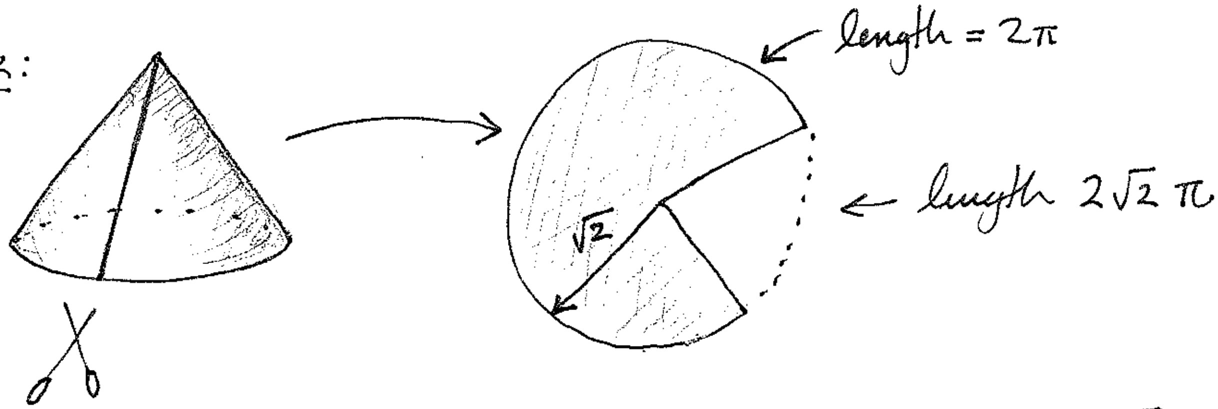
$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -v\sin u & v\cos u & 0 \\ \cos u & \sin u & 1 \end{vmatrix} = (v\cos u, v\sin u, -v)$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{v^2\cos u + v^2 \sin u + v^2} = \sqrt{2}\, v.$$

<u>Area:</u> $\iint_S 1\, dA = \iint_D |\vec{r}_u \times \vec{r}_v|\, du\, dv = \int_0^1 \int_0^{2\pi} \sqrt{2}\, v\, du\, dv$

$= \boxed{\sqrt{2}\,\pi.}$

Check:



$$\Rightarrow \text{Area}(\triangle) = \frac{1}{\sqrt{2}}\,\text{Area}(\bigcirc) = \frac{1}{\sqrt{2}}\,\pi(\sqrt{2})^2 = \boxed{\sqrt{2}\,\pi}\ \checkmark.$$

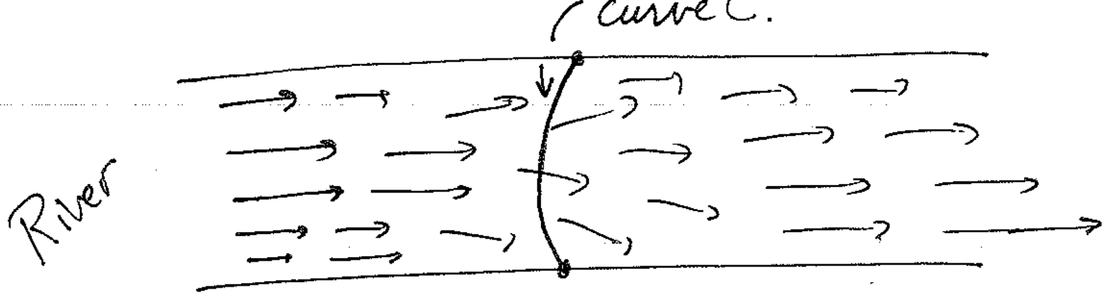
Integral: $$\iint_S xy+z\ dA = \int_0^1\int_0^{2\pi}(v^2\underbrace{\sin u\cos u}_{\frac{d}{du}\frac{1}{2}\sin^2 u \text{ or } \frac{1}{2}\sin 2u}+v)\sqrt{2}v\,du\,dv$$

$$= \int_0^1 \left(v^2\sin^2 u + vu\right)\Big|_{v=0}^{v=2\pi}\sqrt{2}v\,dv = \int_0^1 2\pi\sqrt{2}\,v^2\,dv$$

$$= 2\sqrt{2}\pi\frac{v^3}{3}\Big|_{v=0}^{1} = \frac{2\sqrt{2}\pi}{3}$$

Average: $$= \frac{1}{\text{Area}}\iint_S f\,dA = \frac{2}{3}.$$

Integrating vector fields: Back to $\mathbb{R}^2$.

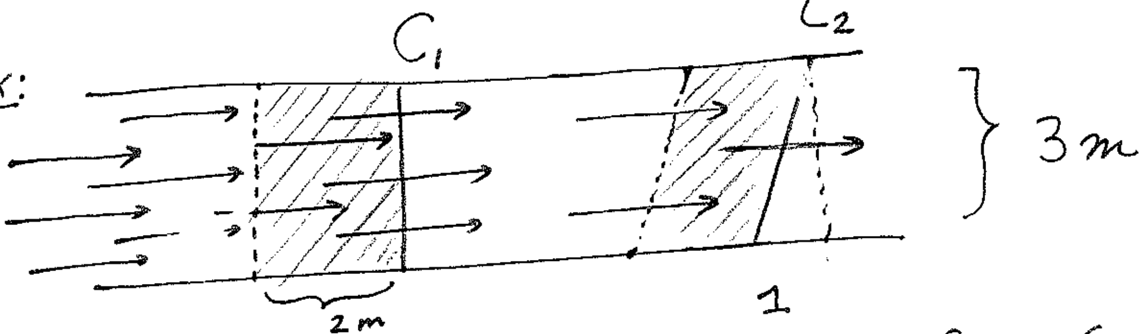


Q: What is the rate that fluid is crossing C? (The Flux.)

$\vec{F}: \mathbb{R}^2 \to \mathbb{R}^2$ rep. fluid flow

Note: Answer will have units (area)/(time) since this is a 2^d-flow.

Ex:



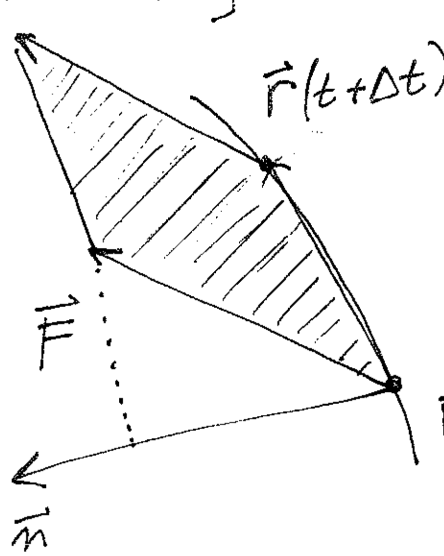
$\vec{F} = (2\,m/s, 0)$

Flux across $C_1 = 6\ m^2/s$

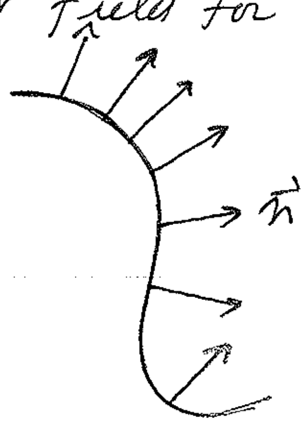
Flux across $C_2 = 6\ m^2/s$

[Q: Should this be more or less than with C_1?]

Closeup of a general curve, where $\vec{F}$ is essentially constant



Let $\vec{n}$ be a unit normal vector field for C.

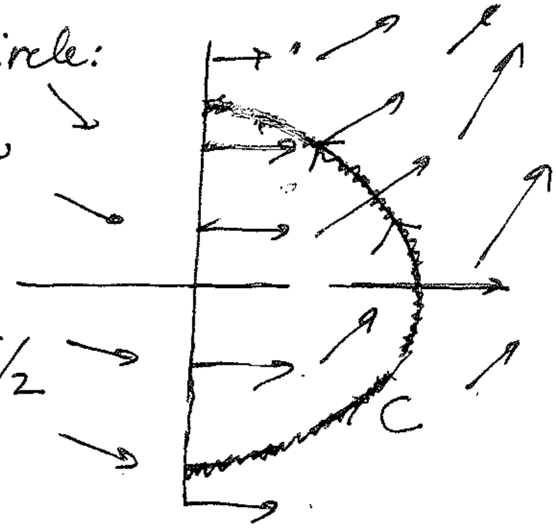


Area of fluid crossing segment.

$\approx (\vec{F} \cdot \vec{n})\,|\vec{r}\,'(t)|\,\Delta t.$

Hence $\text{Flux} = \displaystyle\int_C \overbrace{(\vec{F} \cdot \vec{n})}^{\text{function along the curve.}}\, ds$

$\underline{Ex}$: $\vec{F} = (1, x)$ and $C = \frac{1}{2}$ unit circle:



<u>Parameterization</u>:

$$\vec{r}(t) = (\cos t, \sin t) \quad -\pi/2 \leq t \leq \pi/2$$

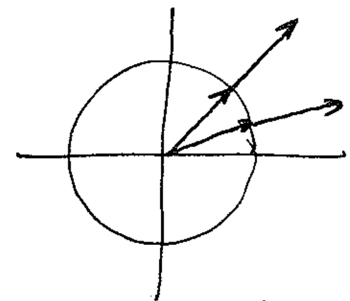
$$\vec{n}(t) = (\cos t, \sin t)$$

Special to this situtation!

<u>Check</u>:

$$\vec{n}(t) \cdot \vec{r}\,'(t) = (\cos t, \sin t) \cdot (-\sin t, \cos t)$$
$$= 0.$$

$$|\vec{n}(t)| = 1.$$



<u>Flux</u>:

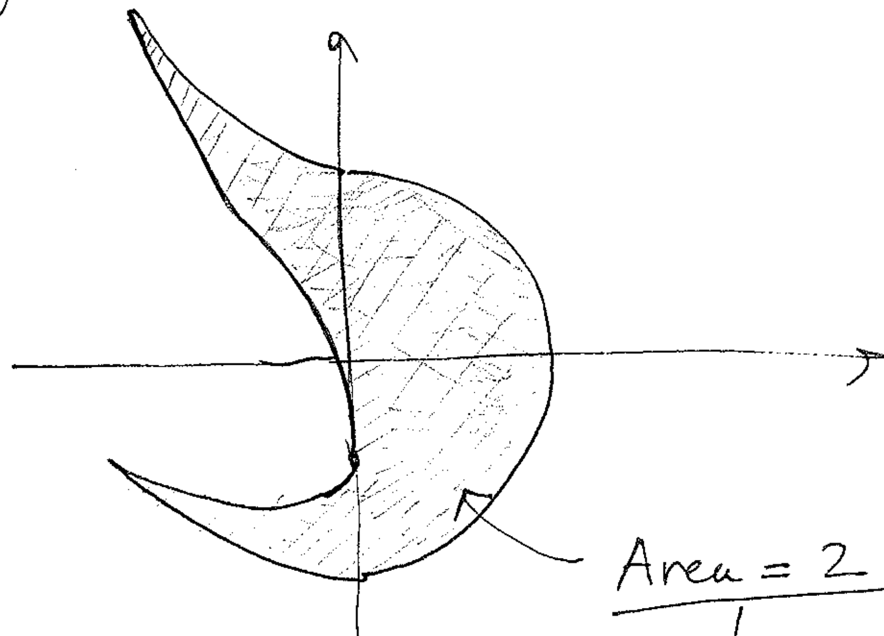
$$\int_C (\vec{F} \cdot \vec{n})\, ds = \int_{-\pi/2}^{\pi/2} \vec{F}(\vec{r}(t)) \cdot \vec{n}(t) \, \overbrace{\underbrace{|\vec{r}\,'(t)|}_{1}\, dt}^{ds}$$

$$= \int_{-\pi/2}^{\pi/2} (1, \cos t) \cdot (\cos t, \sin t)\, dt$$

$$= \int_{-\pi/2}^{\pi/2} \cos t + \sin t \cos t \, dt = 2$$

What does the region of water look like which flows across C in one unit of time?

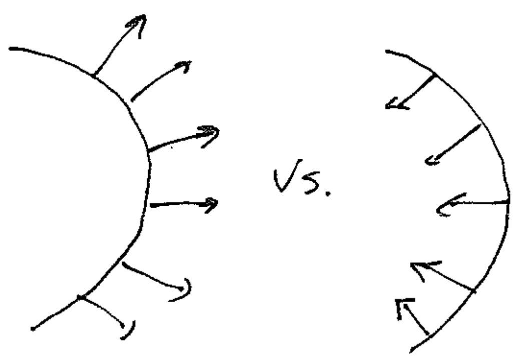
A.



Note to self: Since $\operatorname{div} \vec{F} = 0$.

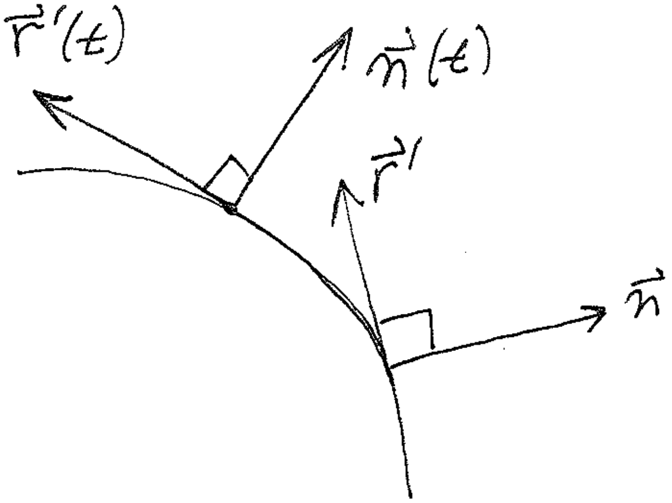
Show animation of this

Note: When you compute the flux, must choose a direction for $\vec{n}$:



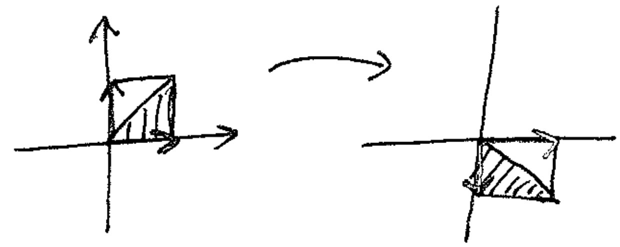
Method for computing $\vec{n}$:

Assume $\vec{r} : [a,b] \longrightarrow C$ has unit speed (e.g. param by arc length).



Relation between:

rotate by 90°. $\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$



$$T(u,v) = (v,-u)$$

So if $\vec{r}(t) = (r_1(t), r_2(t))$ have

$$\vec{n}(t) = (+r_2'(t), -r_1'(t))$$

Now suppose $\vec{F} = (F_1, F_2)$. Then

$$\int_C (\vec{F}\cdot\vec{n})\,ds = \int_a^b (F_1(\vec{r}(t)), F_2(\vec{r}(t)))\cdot(r_2'(t), -r_1'(t))\,\overbrace{dt}^{\text{unitspeed} = ds}$$

$$= \int_a^b \vec{F}_1(\vec{r}(t))\,r_2'(t) - \vec{F}_2(\vec{r}(t))\,\vec{r}_1'(t)\,dt$$

$$= \int_C \vec{G}\cdot d\vec{r} \quad \text{where } \vec{G} = (-F_2, F_1)$$

$$= \iint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\right) dA$$

What does this measure?

Green's Thm, assuming $C = \partial D$